Chapter 1 Intro What are the differences between Pre-Calc & Calculus?

→ Pre-Calc gave us a basis for more general topic like trigonometry, vectors, and Polar-graphing; Calculus is focused on continuous or instantaneous change

Pre-Calc
→ Areas of basic shapes

Calc
→ Changes in Rates
→ Integral – Area of irregular shapes

Aug. 28. 2024

What is Calculus?

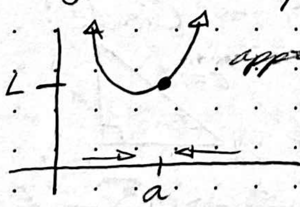
→ Branch of Math, deals with finding derivatives & integrals of functions
↳ Based on methods of summation of infinitesimal differences (smaller & smaller)

2 Branches
→ Differential → slopes, rates of change, derivatives, etc.
→ Integral → areas, volumes, integrals, etc.
} Limits are used in both branches

1.2 Finding Limits Graphically

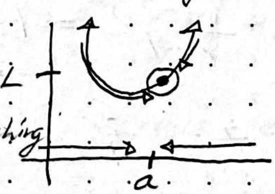
Limit – y-value the function tends to approach from both sides of the x-axis

Ex:



approaching $L$ → $\lim_{x \to a} f(x) = L$

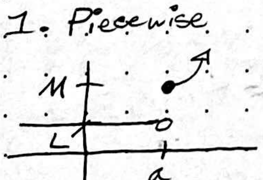
↳ limit notation is true for approaching the limit from both sides x-axis



When does a limit not exisist?

DNE = does not exist
↳ Must state a reason

1. Piecewise



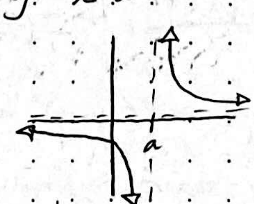
$\lim_{x \to a} f(x) = DNE$

$\left.\begin{array}{l}\lim_{x \to a^+} f(x) = M \\ \lim_{x \to a^-} f(x) = L\end{array}\right\} \neq$

only with jump discontinuity

2. Unbounded behavior

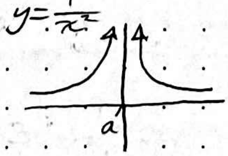
$y = \frac{1}{x-a}$



$\lim_{x \to a} f(x) = DNE$, because of unbounded behavior

$\lim_{x \to a} f(x) = DNE$, because $\underset{\infty}{\lim_{x \to a^+} f(x)} \neq \underset{-\infty}{\lim_{x \to a^-} f(x)}$

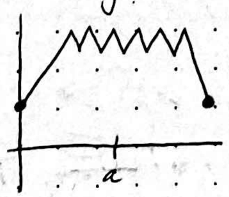
$y = \frac{1}{x^2}$



$\lim_{x \to} f(x) = DNE$ due to unbounded behavior

$\lim_{x \to x} f(x) = \infty$

3. Oscillating Behavior



$\lim_{x \to a} f(x) = DNE$ due to oscillation

How to find a limit: (methods)

1. Direct substitution → plug in $x$ to $f(x)$.
2. Use factoring
3. Use rationalization techniques (conjugates)
*4. Use graphing
*5. Use a table

* = When using calculator

Ex 1: Evaluate $f(x) = \frac{x}{(\sqrt{x+1}-1)}$ @ several x-values near 0 & estimate limit

$\rightarrow \frac{0}{(\sqrt{0+1}-1)} \rightarrow \frac{0}{0} \rightarrow$ DNE, cannot divide by 0; indeterminate

Table:

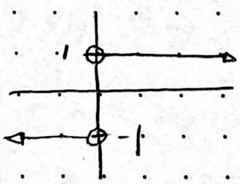
| $x$ | -0.1 | -0.01 | -0.001 | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 1.9487 | 1.995 | 1.9995 | ind | 2.0005 | 2.005 | 2.000 |

$\lim_{x \to 0} f(x) = 2$

Ex. 2: Find limit of $f(x)$ as $x$ approaches $\quad f(x) = \begin{cases} 1, & x \neq 2 \\ 0, & x = 2 \end{cases}$

$\lim_{x \to 2} f(x) = 1$ $\quad f(2) = 0$

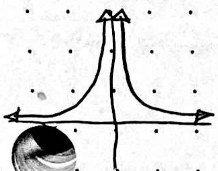
Ex 3: Show that limit $\lim_{x \to 0} \frac{|x|}{x}$ does not exist



$\lim_{x \to 0^+} \frac{|x|}{x} = 1$ $\quad \lim_{x \to 0} \frac{|x|}{x} =$ DNE, because $\lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$

$\lim_{x \to 0^-} \frac{|x|}{x} = -1$

Ex 4: Limit $\lim_{x \to 0} \frac{1}{x^2}$ $\quad \lim_{x \to 0} \frac{1}{x^2} = \infty$



$\lim_{x \to 0} \frac{1}{x^2} =$ DNE, due to unbounded behavior

HW: p. 55 Qs: 3, 7, 11-19, 21-23, 49-52, 59.

1.3 Warm-Up

$\lim_{x \to 3} \frac{x-3}{x^2-9}$

| $x$ | 2.9 | 2.99 | 2.999 | 3 | 3.001 | 3.01 | 3.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.1695 | 0.1689 | 0.1667 | ? | 0.1666 | 0.1664 | 0.1639 |

$\lim_{x \to 3} \frac{x-3}{x^2-9} = 0.1666$ $\rightarrow$ the limit exists

3 acceptable reasons for DNE: ① $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

$\frac{n}{0} \rightarrow$ undefined $\quad$ ② Oscillating behavior

$\frac{0}{0} \rightarrow$ indeterminate $\quad$ ③ Unbounded / Boundless behaviour

~~1.3 HW p67 Qs: 3, 14, 15, 23, 28, 31, 37-45 odd, 51, 55, 57, 59, 63, 64, 73, 85, 86~~

1.3 Eval. limits analytically Aug 27, 2024

$\lim_{x \to c} f(x)$ does not depend on value of $f$ at $x=c$

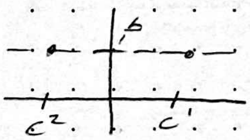
Sometime $f(c)$ is

Use substitution first for any limit problem involving a function

Theorem 1.1 Basic Limits

let $b$ & $c$ be real numbers & $n$ a positive integer.

1. $\lim_{x \to c} b = b$ 2. $\lim_{x \to c} x = c$ 3. $\lim_{x \to c} x^n = c^n$



Identity function Direct substitution

Ex 1: a: $\lim_{x \to 4} 5 = 5$ b: $\lim_{x \to -2} x = -2$ c. $\lim_{x \to 3} x^2 = 9$

Theorem 1.2 Properties of Limits

$b$ & $c$ real numbers, $n$ positive integer, $f$ & $g$ functions with these

limits: $\lim_{x \to c} f(x) = L$ $\lim_{x \to c} g(x) = K$ $b \underbrace{\lim_{x \to c} f(x)}_{L} \to bL$

1. Scalar multiple $\lim_{x \to c} [bf(x)] = bL$
2. Sum or difference $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$
3. Product $\lim_{x \to c} [f(x)g(x)] = LK$
4. Quotient $\lim_{x \to c} \left[\frac{f(x)}{g(x)}\right] = \frac{L}{K}$ if $K \neq 0$
5. Power $\lim_{x \to c} [f(x)]^n = L^n$

Ex 2: Use properties to evaluate following:

a) $\lim_{x \to 2} (3x^2 - 1) \to 3 \lim_{x \to 2} x^2 - \lim_{x \to 2} 1 \to 3 \cdot 4 - 1 \to \boxed{11}$

Use direct substitution for all kinds of functions.

Ex 3: $\lim_{x \to 1} \frac{x^2 + x + 2}{x + 1} \to \frac{1+1+2}{1+1} \to \frac{4}{2} \to \boxed{2}$

Ex 4: a) $\lim_{x \to 0} \sqrt{x^2 + 4}$

$\to \sqrt{0^2 + 4}$

$\to \sqrt{4}$

$\to 2$

b) $\lim_{x \to 3} \sqrt[3]{2x^2 - 10}$

$\to \sqrt[3]{2(3^2) - 10}$

$\to \sqrt[3]{2(3^2) - 10}$

$\to \sqrt[3]{8}$

$\to 2$

Ex 5:

ⓐ $\lim_{x\to 0} \tan x \to \tan 0 \to \frac{0}{1} \to \boxed{0}$

ⓑ $\lim_{x\to \pi} (x \cos \pi) \to \pi . \cos \pi \to \pi(-1) \to \boxed{-\pi}$

ⓒ $\lim_{x\to 0} \sin^2 x \to (\sin 0)(\sin 0) \to 0 \cdot 0 \to \boxed{0}$

Ex 6:

$\lim_{x\to 1} \frac{x^3-1}{x-1} \to \frac{1^3-1}{1-1} \to \frac{0}{0} \to$ undetermined / indeterminate

$\lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{(x-1)} \to \lim_{x\to 1} (x^2+x+1)$

$\to 1^2+1+1 \to \boxed{3}$

$$\boxed{\begin{aligned} a^3-b^3 &= (a-b)(a^2+ab+b^2) \\ a^3+b^3 &= (a+b)(a^2-ab+b^2) \end{aligned}}$$

Ex 7:

$\lim_{x\to -3} \frac{x^2+x-6}{x+3} \to \frac{9+3-6}{-3+3} \to \frac{0}{0}$ = indeterminate.

$\lim_{x\to -3} \frac{(x+3)(x-2)}{x+3} \to \lim_{x\to -3} x-2 \to -3-2 \to \boxed{-5}$

↳ Synthetic division

$$\begin{array}{r|rrr} -3 & 1 & 1 & -6 \\ & \downarrow & -3 & 6 \\ \hline & 1 & -2 & 0 \end{array} \to (x-2)$$

Ex 8: $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} \to \frac{\sqrt{0+1}-1}{0} \to \frac{0}{0}$ indeterminate

$= \lim_{x\to 0} \left(\frac{\sqrt{x+1}-1}{x}\right)\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right) \to x+1 \cancel{-\sqrt{x+1}+\sqrt{x+1}} -1 \to \frac{x}{x(\sqrt{x+1}+1)}$

$\to \frac{1}{\sqrt{x+1}+1} \to \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1} \to \frac{1}{\sqrt{1}+1} \to \boxed{\frac{1}{2}}$

Theorem 1.8. Squeeze Theorem / Sandwich Theorem

If $h(x) \le f(x) \le g(x)$ in interval containing $c$, exept $c$ itself,

and if $\lim_{x\to c} h(x) = L = \lim_{x\to c} g(x)$, then $\lim_{x\to c} f(x)$ exists & equals $L$

Two Special Trig. Limits

$\lim_{x\to 0} \frac{\sin x}{x} = 1$ & $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

$\frac{\sin a(x)}{a(x)} = 1$

Ex. 9: $\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \left(\frac{\sin x}{\cos x}\right)\left(\frac{1}{x}\right) \to \left(\lim_{x\to 0} \frac{1}{\cos x}\right)\left(\lim_{x\to 0} \frac{\sin x}{x}\right)$

$\left(\frac{1}{1}\right)(1) \to \boxed{1}$

Learn: Intercepts, domain, solving for x with trig, trinomial factoring solving triangles trig

1.3 HW. p.67 Q. 3, 14, 15, 23, 28, 31, 37-45 odd, 51, 55, 57, 59, 63, 64, 73, 85, 86.

1.4 Warm up. ⓐ $\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1} \to \left(\frac{(\sqrt{x+3}-2)}{(x-1)}\right)\left(\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}\right)$

$\to \frac{x+3-4}{(x-1)(\sqrt{x+3}+2)} \to \frac{x-1}{(x-1)(\sqrt{x+3}+2)} \to \frac{1}{\sqrt{x+3}+2} \to \frac{1}{\sqrt{1+3}+2} \to \boxed{\frac{1}{4}}$

ⓑ $\lim_{x\to c} f(x) = 3 \quad \lim_{x\to c} g(x) = 2$ a) 10 b) 5 c) 6 d) $\frac{3}{2}$

$\lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \to$ (85) $f(x) = x^2 - 4x$

$\to \lim_{\Delta x\to 0} \frac{(x+\Delta x)^2 - 4(x+\Delta x) - (x^2-4x)}{\Delta x} \to \lim_{\Delta x\to 0} \frac{\cancel{x^2}+2x\Delta x+\Delta x^2 \cancel{-4x} - 4\Delta x \cancel{-x^2} \cancel{+4x}}{\Delta x}$

$\to \lim_{\Delta x\to 0} \frac{\Delta x(2x+\Delta x-4)}{\Delta x} \to 2x+\Delta x-4 \to 2x+0-4 \to \boxed{2x-4}$

(86)

$f(x) = \sqrt{x} \quad \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \to f(x) = \lim_{\Delta x\to 0} \frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}$

$\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}\left(\frac{\sqrt{x+\Delta x}+\sqrt{x}}{\cancel{\Delta x}}\right) \to \frac{\cancel{x}+\Delta x-\cancel{x}}{\Delta x(\cdots)} \to \frac{\Delta x}{\Delta x\, \sqrt{x+\Delta x}+\sqrt{x}}$

$\to \frac{1}{\sqrt{\Delta x+x}+\sqrt{x}} \to \frac{1}{2\sqrt{x}}$

(64) $\lim_{x\to 0} \frac{3(1-\cos x)}{x} \to 3\left(\lim_{x\to 0} \frac{1-\cos x}{x}\right) \to \frac{1-\cos x}{x} = 0 \cdot 3 = \boxed{0}$

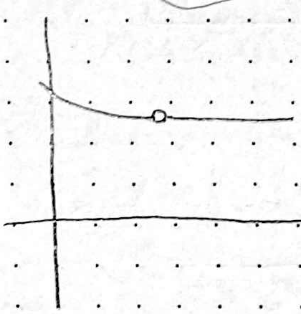
1.4 Continuity & 1 directional limits

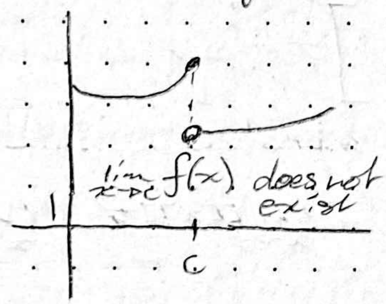
A function is continuous if:

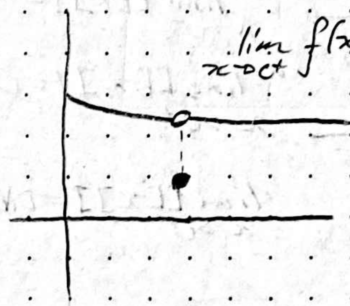
→ there are no jumps → It is always defined

→ there is no hole → there are no vertical asymptotes

$\frac{(x+3)}{(x+3)(x-3)}$ Hole → Vertical asymptote






Continuity at a point:

Function $f$ is continuous at c if:

1 $f(c)$ is defined

2 $\lim_{x \to c} f(x)$ exists

3 $\lim_{x \to c} f(x) = f(c)$

at an open interval:

if continuous at each point $(-\infty, \infty)$ everywhere continuous

2 types of discontinuity

1. Non-removable → jump & ver. asym. (infinite disc.)
2. Removable → Hole

} if you don't know, use Non-removable discontinuity

Ex:

ⓐ $f(x) = \frac{1}{x}$ $x \neq 0$

$f(x)$ is continuous on $(-\infty, \infty)$, except at $x = 0$, where there is a non-removable infinite discontinuity.

ⓑ $g(x) = \frac{x^2 - 1}{x - 1}$

$g(x)$ is continuous on $(-\infty, \infty)$, except at $x = 1$, where there is a removable discontinuity

ⓒ $h(x) = \begin{cases} x+1, & x \leq 0 \\ x^2+1, & x > 0 \end{cases}$ $\underbrace{\lim_{x \to 0^-} f(x) = 1 \quad \lim_{x \to 0^+} f(x) = 1}_{\text{equal limits}}$ $h(1) = 1$

$h(x)$ is continuous on $(-\infty, \infty)$.

ⓓ $y = \sin x$

The function $y = \sin x$ is everywhere continuous.

<u>One-sided limits</u> → they exist (no DNE)

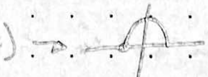
from right → +

from left → −

$\lim_{x \to 0^+}$ → limit as $x$ approaches from the right

$\infty$ or $-\infty$ work as answers, never DNE

Ex 2: $f(x)=\sqrt{4-x^2}$ $\lim\limits_{x\to -2^+} f(x)=0$

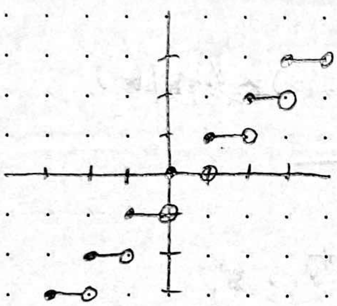
$4-x^2 \to$  $f(x) \to$ 

Ex 3: Greatest Integer Function $\to$ Biggest Integer $\leq x$ in. $[[x]]$

$\to$ round down to nearest integer

Evaluate:

a) $[[1]]=1$ b) $[[-4]]=-4$ c) $[[1.3]]=1$ d) $[[-3.4]]=-4$



$\left.\begin{aligned}\lim_{x\to 0^-}[[x]]&=-1\\ \lim_{x\to 0^+}[[x]]&=0\end{aligned}\right]\neq$

$\lim\limits_{x\to 0}[[x]]=DNE,$ $\lim\limits_{x\to 0^-}[[x]] \neq \lim\limits_{x\to 0^+}[[x]]$

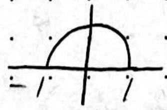
Existance of a limit:

$\lim\limits_{x\to c} f(x)=L$ only if $\lim\limits_{x\to c^-} f(x)=L$ $\underline{\underline{\text{and}}}$ $\lim\limits_{x\to c^+} f(x)=L$

Continuity on closed interval $[a,b]$

when $f$ is cont. on $(a,b)$ and $\lim\limits_{x\to a^+} f(x)=f(a)$

and $\lim\limits_{x\to b^-} f(x)=f(b)$ $\to \underbrace{\text{from right at } a \text{ \& at } b \text{ from left}}_{\text{continuous}}$

Ex 4: $f(x)=\sqrt{1-x^2}$  $[-1,1]$ $\lim\limits_{x\to 1^-} f(x)=0$ continuous at $(-1,1)$

$\lim\limits_{x\to -1^+} f(x)=0$

Determine if $f(x)$ is cont:

Ex 6: a) $f(x)=x+\sin x$

$f(x)$ is everywhere cont.

b) $f(x)=3\tan x$ $x\neq \frac{\pi}{2}+n\pi$

$f(x)$ is continuous on

$\ldots\left(-\frac{\pi}{2},\frac{\pi}{2}\right),\left(\frac{\pi}{2},\frac{3\pi}{2}\right),\left(\frac{3\pi}{2},\frac{5\pi}{2}\right)\ldots$

c) $f(x)=\frac{x^2+1}{\cos x}$

$\to$ cont. @ $(-\infty,\infty)$

exept @ $\cos x=0$

and $x=\frac{\pi}{2}+\pi n$

$\to f(x)$ is cont. on $(-\infty,\infty)$

exept at $x=\frac{\pi}{2}+n\pi$

Ex 7: Determine if cont.

a) $f(x)=\tan x$

$(-\infty,\infty)$, exept @ $x=\frac{\pi}{2}+\pi n$

b.

1.4 HW p.79 3, 6, 9, 12, 14, 15, 17, 23, 25, 27, 28, 30, 43, 47, 51, 53,

61, 95

1.5 Double check 1.4 HW

Warm-up: If continuous. If not, find $x$-axis location for discont:

1. $f(x) = \begin{cases} \frac{x}{2} + \frac{5}{2}, & x \le 0 \\ 2x+1, & x > 0 \end{cases}$ jump

Non removable discontinuity @ $x = 0$, everywhere else continuous.

2. $f(x) = \frac{-x^2 - 2x + 3}{x+3} \to \frac{(\cancel{x+3})(-x-1)}{\cancel{x+3}} \to \frac{-x-1}{1} \to x \neq -3$

Removable discontinuity @ $x = -3$, everywhere else continuous.

Intermediate Value Theorem Practice

(96) $f(x) = x^2 - 6x + 8$, $[0,3]$, $f(c) = 0$

$0 = x^2 - 6x + 8 \to (x-4)(x-2) = 0 \to x = 2, 4$ $[0,3]$

$\boxed{c = 2}$ 4 is not $[0,3]$

(97) $f(x) = x^3 - x^2 + x - 2$, $[0,3]$, $f(c) = 4$

$0 = 1x^3 - x^2 + x - 6$ Use $\frac{p}{q}$ p = factor of constant, q = factor of leading coefficient

$p = \pm 1, \pm 2, \pm 3$
$q = \pm 1$

$$\begin{array}{r|rrrr} 3 & 1 & -1 & 1 & -6 \\ & & 3 & 6 & 7 \\ \hline & 1 & 2 & 7 & 1 \end{array} \qquad \begin{array}{r|rrrr} 1 & 1 & -1 & 1 & -6 \\ & & 1 & 0 & 1 \\ \hline & 1 & 0 & 1 & 1 \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & -1 & 1 & -6 \\ & & 2 & 2 & 6 \\ \hline & 1 & 1 & 3 & 10 \end{array}$$

$C = 2$ because 2 is $[0,3]$

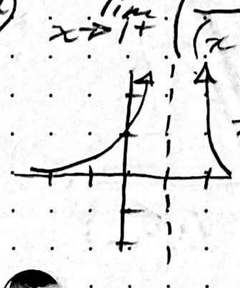
## 1.5 Limits at Infinity Sep. 3. 2024

HW. p88 Q: 15, 17, 23, 27, 39, 43, 55, 56, 65-68.

Ex 1: Determine lim of function $x \to 1$ from left & right

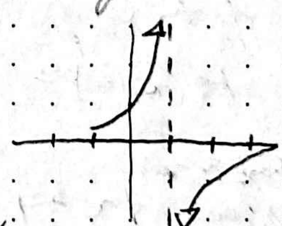
a) $\lim_{x \to 1^+} \left( \frac{1}{(x-1)^2} \right) = \infty$

$\lim_{x \to 1^-} \left( \frac{1}{(x-1)^2} \right) = \infty$



b) $\lim_{x \to 1^+} f(x) = -\infty$

$\lim_{x \to 1^-} f(x) = \infty$



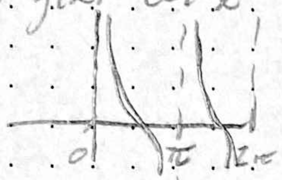
Verticle line $f(x)$ approaches but never touches
- Pre-Calc Verticle Asymptote

Calc: If $f(x)$ approaches infinity (or negative ") as $x$ approaches $c$ from left or right then $x = c$ is a verticle asymptote

Ex 2:

(a) V.A. is @ $x=-1$  (b) $\frac{x^2+1}{x^2-1} \to$ V.A is @ $x=-1, 1$

(c) $f(x)=\cot x$  V.A. is @ $x=\pi n$



Ex 3: $f(x) = \frac{x^2+2x-8}{x^2-4} \to \frac{(x+4)\cancel{(x-2)}}{\cancel{(x-2)}(x+2)} \to$ V.A. is @ $x=-2$

Ex 4: $\frac{x^2-3x}{x-1} \to \frac{x(x-3)}{x-1}$  $x=1$ is V.A.

$\lim_{x\to 1^+} f(x) = -\infty$  $\lim_{x\to 1^-} f(x) = \infty$

| $x$ | 2 | 1.5 |
|---|---|---|
| $y$ | -2 | -4.5 |

Ex 5: (a) $\lim_{x\to 0}\left(1+\frac{1}{x^2}\right) \to 0+\infty = \infty$

(b) $\lim_{x\to 1} \frac{x^2+1}{\cot \pi n} = \frac{2}{-\infty} \to 0$  $\lim_{x\to 1} x^2+1 \to 2$  $\lim_{x\to 1} \cot \pi n \to -\infty$

(c) $\lim_{x\to 0^+} 3\cot x = \infty$

HW Questions 1.4

(14) $\lim_{x\to 10^+} \frac{|x-10|}{x-10} \to$ from right: $\frac{x-10}{x-10}=1$  from left $\frac{-(x-10)}{x-10}=-1$

$\to 1$ because $x\to 10^+ \to$ from right

(15) $\lim_{\Delta x\to 0^+} \frac{\left(\frac{x}{x}\right)\frac{1}{x+\Delta x} - \frac{1}{x}\left(\frac{x+\Delta x}{x+\Delta x}\right)}{\Delta x} \to \lim_{\Delta x\to 0^+} \frac{\frac{x-x-\Delta x}{x(x+\Delta x)}}{\Delta x} \to \frac{-\Delta x}{x(x+\Delta x)} \cdot \frac{1}{\Delta x}$

$\to \frac{-\Delta x}{x(x+\Delta x)(\Delta x)} \to \frac{-1}{x(x+\Delta x)} \to \frac{-1}{x(x+0)} \to \frac{-1}{x^2}$

Hypothesis: if you have a fraction with abs$(x+a)$ look @ if $x\to b$ from left or right. If from right, abs$(x+a) = (x+a)$. if left, abs$(x+a) = -(x+a)$

Ex:

$\lim_{x\to 5^+} \frac{|x-5|}{(x-5)} \to \lim_{x\to 5^+} \frac{x-5}{x-5} = 1$  if $\frac{\text{trig. func}(ax)}{ax} = a$

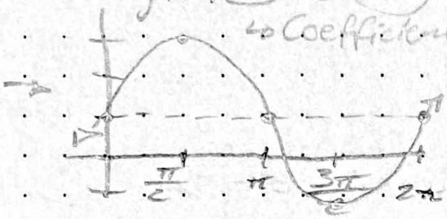
$\to \lim_{x\to 5^-} \frac{-(x-5)}{x-5} = -1$

Ex: $\frac{\sin 3x}{3x} = 3$

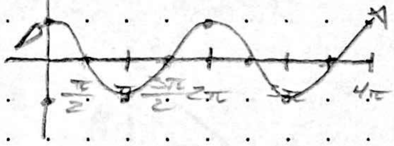
Chapter P Review    Solving Trig Equations

Trig Graphing:

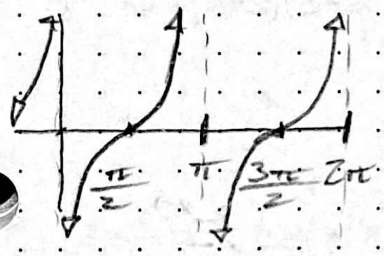
1. $f(x) = 2\sin x + 1$ Seperate from $x$, $k$ value
↳ Coefficient of trig func is amplitude scalar




2. $f(x) = -\cos(x - \pi)$ negative coefficient → $x$-axis flip
↳ constant affecting $x$ before trig. func.
↔ $h$ value




3. $f(x) = \tan(x + \frac{\pi}{2})$




c) $6 - \sin\theta = 2\sin^2\theta + 5$

→ $2\sin^2\theta + \sin\theta - 1 = 0$

↳ $2\sin^2\theta + 2\sin\theta - \sin\theta - 1 = 0$

→ $(2\sin\theta - 1)(\sin\theta + 1) = 0$

$2\sin - 1 = 0$ & $\sin\theta + 1 = 0$

$\sin\theta = \frac{1}{2}$ & $\sin\theta = -1$

when

$\theta = \frac{\pi}{6} + 2\pi n$

$\frac{5\pi}{6} + 2\pi n$

$\theta = \frac{3\pi}{2} + 2\pi n$, $n \in \mathbb{Z}$

Solving Trig Equations:

1 a) $-2\tan\theta\cos\theta + 2\tan\theta = 3\tan\theta$

↳ $-2\tan\theta\cos\theta - \tan\theta = 0$

↳ $\tan\theta(-2\cos\theta - 1) = 0$

$\tan\theta = 0$ when $\theta = \pi n$, $\in \mathbb{Z}$

$-2\cos\theta = 1$ → $\cos\theta = -1/2$

→ $\cos\theta = -\frac{1}{2}$ when

$\theta = \frac{2\pi}{3} + 2\pi n$ & $\theta = \frac{4\pi}{3} + 2\pi n$, $n \in \mathbb{Z}$

b) $2\cot\theta + 1 = \cot^2\theta + 2$

↳ $\cot^2\theta - 2\cot\theta + (2-1) = 0$

↳ $\cot^2\theta - 2\cot\theta + 1 = 0$

→ $x = \cot\theta$ → $x^2 - 2x + 1 = 0$

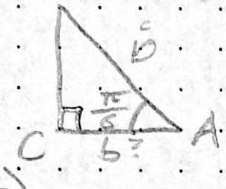
→ $(x-1)(x-1) = 0$ or $(x-1)^2 = 0$

$x - 1 = 0$ → $x = 1$ → $\cot\theta = 1$

$\cot\theta = 1$ when $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

Right Triangle Trig

a). C is 90°   find b if $A = \frac{\pi}{6}$ & c=10



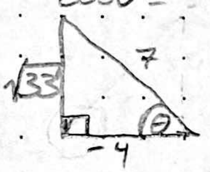

$\cos\frac{\pi}{6} = \frac{b}{10} \to \frac{\cos\frac{\pi}{6}}{10} = b \to \frac{\sqrt{3}}{2}(10) = b$

$\to 5\sqrt{3} = b$

b)

if $\frac{\pi}{2} < \theta < \pi$ & $\sec\theta = \frac{-7}{4}$, what is $\tan\theta$ & $\sin 2\theta$

$\cos\theta = -\frac{4}{7}\ \frac{opp}{hyp}$




$16 + b^2 = 49$

$b^2 = 33$

$\sin\theta = \frac{\sqrt{33}}{7}$

$\tan\theta = \frac{\sqrt{33}}{7} \cdot -\frac{7}{4}$

$\to -\frac{\sqrt{33}}{4}$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\to 2\left(\frac{\sqrt{33}}{7}\right)\left(-\frac{4}{7}\right)$

$\to -\frac{8\sqrt{33}}{49}$

Points of Intersection

$x + y = 1 \to y = -x + 1$   $y = -0 + 1 \to 1$   $y = -3 + 1 \to -2$

$y = -x^2 + 2x + 1 \to -x + 1 = -x^2 + 2x + 1 \to x^2 - 3x + 0 \to (x-3)(x+0) = 0$

$x = 0, 3 \to (0,1), (3,-2)$

Transformation

$f(x) = \sqrt{x} \to f(x) = \underbrace{-2}_{x\text{-flip amp}}\sqrt{\underbrace{x+3}_{\text{h}}} + \frac{3}{2} \to k$

• Vert. streched   • flipped on x-axis   • left 3   • up 3/2

Even-Odd funcs.

if $f(x) = -x^2 + 8x - 16$ is even or odd, explain:

Neither, not y or origin symmetry

if $f(x) = \sqrt[3]{x}$ is even or odd, explain:

Odd, origin-sym. : $f(-x) = \sqrt[3]{-x} = -\sqrt[3]{x}$

Symmetry   How do you determine the symmetry of a graph of a function?

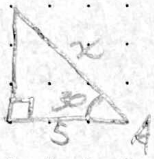
$f(-x) = f(x) \to$ even - y-axis sym.

$f(-x) = -f(x) \to$ odd - origin-sym.

if the function is consistent regardless of if $[x(-1)]$ or $(-1)[x]$, the function is odd - origin sym.

Otherwise, it's even - y-axis sym.

$A=30^\circ$ $a=5$ find hyp

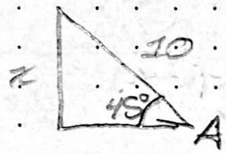



$\cos 30 = \frac{5}{x} \to \frac{\sqrt{3}}{2} = \frac{5}{x}$

$\to \frac{x}{1}\cdot\frac{\sqrt{3}}{2} = 5 \to \frac{x\sqrt{3}}{2} = 5 \to x\sqrt{3} = 10$

$\to x = \frac{10}{\sqrt{3}} \to \frac{10\sqrt{3}}{3}$

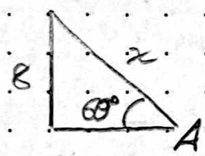
$A=45^\circ$ hyp$=10$, find opp




$\sin 45^\circ = \frac{x}{10} \to \frac{\sqrt{2}}{2} = \frac{x}{10} \to \frac{10}{1}\cdot\frac{\sqrt{2}}{2} \to$

$x = 5\sqrt{2}$

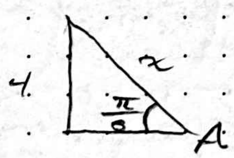
$A=60^\circ$ opp$=8$ find hyp




$\sin 60 = \frac{8}{x} \to x\sin 60 = 8 \quad \frac{x}{1}\left(\frac{\sqrt{3}}{2}\right) \to \frac{x\sqrt{3}}{2} = 8$

$\to x\sqrt{3} = 16 \to x = \frac{16}{\sqrt{3}} \to \frac{16\sqrt{3}}{3}$

$A=\frac{\pi}{6}$ opp$=4$ find hyp




$\sin\frac{\pi}{6} = \frac{4}{x} \to \frac{1}{2} = \frac{4}{x} \to \frac{x}{2} = 4 \to x = 8$

$2\sin\theta - 1 = 0 \to \sin\theta = \frac{1}{2}$ when $\theta = \frac{\pi}{6} + 2\pi n$, $\theta = \frac{5\pi}{6} + 2\pi n$, $n \in \mathbb{Z}$

interval $[0, 2\pi)$: $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$2\cos^2\theta - 3\sin\theta = 0 \to 2(1-\sin^2\theta) - 3\sin\theta = 0$

$\to 2 - 2\sin^2\theta - 3\sin\theta = 0 \to -2\sin^2\theta - 3\sin\theta + 2 = 0$

$\to 2\sin^2\theta + 3\sin\theta - 2 = 0 \to (2\sin\theta - 1)(\sin\theta + 2) = 0$ $\quad -4$

$\sin\theta = \frac{1}{2}, -2$ when $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$, General: $\left.\begin{array}{l}\theta = \frac{\pi}{6} + 2\pi n \\ \frac{5\pi}{6} + 2\pi n\end{array}\right\} n \in \mathbb{Z}$

$\to$ no solution

$\sin(2\theta) + \sqrt{3}\cos(\theta) = 0 \to 2(\sin\theta)(\cos\theta) + \sqrt{3}\cos\theta = 0$

$\to \cos\theta(2\sin\theta + \sqrt{3}) = 0$

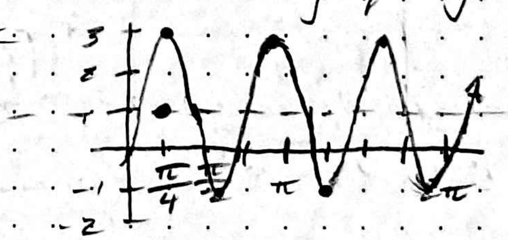
$\cos\theta = 0 \qquad \sin\theta = -\frac{\sqrt{3}}{2}$

when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ & when $\theta = \frac{5\pi}{3}, \frac{4\pi}{3}$

$f(x) = 2\sin\left(3x - \frac{\pi}{4}\right) + 1$

a h k

$\to$ frequency



2.1 Warm-Up Find the limit

$$\lim_{h\to 0}\left\{\frac{\frac{3}{x+h}-\frac{3}{x}}{h}\right\} \to \left(\frac{x}{x}\right)\left(\frac{3}{x+h}\right) - \frac{3}{x}\left(\frac{x+h}{x+h}\right) \to \frac{3x-3x-3h}{x(x+h)} \to \frac{\frac{-3h}{x(x+h)}}{h}$$

Write $\left\{\lim_{x\to 0} \ldots\right\}$ until you evaluate limit!

$$\to \frac{-3h}{x(x+h)}\cdot\frac{1}{h} \to \frac{-3}{x(x+h)} \to \frac{-3}{x^2} = \lim_{h\to 0}\left\{\frac{\frac{3}{x+h}-\frac{3}{x}}{h}\right\}$$

$$\lim_{h\to 0}\frac{\frac{-3h}{x(x+h)}}{h} \to \lim_{h\to 0}\frac{-3h}{x(x+h)}\cdot\frac{1}{h} \to \lim_{h\to 0}\frac{-3}{x(x+h)} = \frac{-3}{x^2}$$

2.1 The Derivative & Tangent Lines

Short Cut to find $f'(x)$ or derivative or slope or rate of change

$f(x)=ax^n \to f'(x)=ax^{n-1}$

ex: 1 $f(x)=x^2+4x$

$f'(x)=2x^1+4x^0 \to 2x+4$ $\quad f'(1)=2(1)+4 \to 6$
$\to$ slope @ $x=1$

What is a tangent?

Straight line that crosses the graph @ $\underline{1}$ point

Secant line?

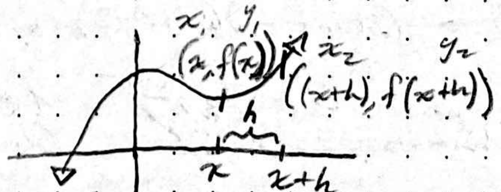
Line touching/crossing graph $\underline{\underline{2}}$ times

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(c+\Delta x)-f(c)}{\Delta x}=m \qquad m = \text{slope}$$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Difference Quotient $\quad h=\Delta x$

$$\frac{f(x+h)-f(x)}{h}$$




Slope form.

$\to \frac{y_2-y_1}{x_2-x_1}$ $\quad m_{sec}=\frac{f(x+h)-f(x)}{x+h-x} \to \frac{f(x+h)-f(x)}{h} \to \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$\hookrightarrow$ makes sec line into tangent

Symbols.

| Function | 1st Derivative | 2nd Derivative | etc |
|---|---|---|---|
| $f(x)$ | $f'(x)$ | $f''(x)$ | $f'''(x)$, $f^{(4)}(x)$, $f^{(5)}(x)$ |
| $y$ | $y'$ or $\frac{dy}{dx}$ | $y''$ or $\frac{d^2x}{d^2x}$ | |

p. 103 Q: 1-4, 7, 9, 17, 21, 23, 29, 33, 37, 39-45

Find slope of tan: 1 find derivative, 2 plug in $x$

Ex 1: $f(x)=2x-3$ when $c=2$

$y=mx+b \to m=2$

Ex 2: $f(x)=x^2+1$ @ $(0,1)$ & $(-1,2)$ [0 & -1]

$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h} \to \lim_{h\to0}\frac{(x+h)^2+1-(x^2+1)}{h}$

$\to \frac{\cancel{x^2}+2xh+h^2\cancel{+1}-\cancel{x^2}\cancel{-1}}{h} \to \frac{2xh+h^2}{h} \to \frac{\cancel{h}(2x+h)}{\cancel{h}} \to 2x+h$

$\lim_{h\to0} \to 2x+0$
$\to 2x$

$f'(x)=2x$ Plug in $x$:

At $(0,1)$, $m=f'(0)=2(0)=0$

At $(-1,2)$, $m=f'(-1)=2(-1)=-2$

Ex 3: Use def. of deriv. (long way)

$f(x)=x^2-2x+1 \to \lim_{h\to0}\frac{f(x+h)-f(x)}{h} \to \lim_{h\to0}\frac{(x+h)^2-2(x+h)+1-(x^2-2x+1)}{h}$

$\to \lim_{h\to0}\frac{\cancel{x^2}+2hx+h^2\cancel{-2x}-2h\cancel{+1}\cancel{-x^2}\cancel{+2x}\cancel{-1}}{h} \to \frac{2hx+h^2-2h}{h}$

$\to \lim_{h\to0}\frac{h(2x+h-2)}{h} \to \lim_{h\to0} 2x+h-2 \to 2x+0-2$

derivative: $2x-2$ $\qquad (1)x^2-2x+1$

Short cut check: $2x-2 \leftarrow 2(1)x-2^0$

b) $f(x)=x^3+2x \qquad f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$

$f'(x)=\lim_{h\to0}\frac{(x+h)^3+2(x+h)-(x^3+2x)}{h}$

$(x+h)^3$

| | | |
|---|---|---|
| $1x^3h^0$ | 1 | $1x^3$ |
| $3x^2h^1$ | 1 2 1 | $3x^2h$ |
| $3x^1h^2$ | 1 3 3 1 | $3xh^2$ |
| $1x^0h^3$ | | $1h^3$ |

$(x+h)^3 = 1x^3+3x^2h+3xh^2+h^3$

$\lim_{h\to0}\frac{\cancel{x^3}+3x^2h+3xh^2+h^3\cancel{+2x}+2h\cancel{-x^3}\cancel{-2x}}{h} \to \lim_{h\to0}\frac{h(3x^2+3xh+h^2+2)}{h}$

$\to \lim_{h\to0} 3x^2+3xh+h^2+2 \to 3x^2+3x(0)+0^2+2 \to \boxed{3x^2+2=f'(x)}$

Ex. 4: $f(x)=\sqrt{x}$ @ $(1,1)$ & $(4,2)$ discuss $f$ at $(0,0)$

$\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$, $\lim_{h\to0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)$

$\to \lim_{h\to0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} \to \lim_{h\to0}\frac{h}{h(\ \ )} \ \lim_{h\to0}\frac{1}{\sqrt{x+h}+\sqrt{x}} \to \frac{1}{\sqrt{x+0}+\sqrt{x}} \to \frac{1}{2\sqrt{x}}$

$m$ at $(1,1) \to f'(1)=\frac{1}{2\sqrt{1}} \to \frac{1}{2}$ $\qquad m$ @ $(4,2) \to f'(4)=\frac{1}{2\sqrt{4}} \to \frac{1}{4}$

2.1 part 2 Sept. 12. 2024

Ex 5: Deriv. with ~~respect~~ to "t" for $y = \frac{2}{t}$

$\rightarrow \frac{dy}{dt} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} \rightarrow \lim_{h \to 0} \frac{\frac{(t)\,2}{(t)\,t+h} - \frac{2}{t}\left(\frac{t+h}{t+h}\right)}{h}$

$\rightarrow \lim_{h \to 0} \frac{\frac{2t}{t(t+h)} - \frac{2t - 2h}{t(t+h)}}{h} \rightarrow \lim_{h \to 0} \frac{\frac{\cancel{2t} - \cancel{2t} - 2h}{t(t+h)}}{\frac{h}{1}}$

$\rightarrow \lim_{h \to 0} \frac{-2h}{t(t+h)} \cdot \frac{1}{h} \rightarrow \lim_{h \to 0} \frac{-2}{t(t+h)} \rightarrow \frac{-2}{t^2} = \frac{dy}{dt}$

Quick Check: $y = \frac{2}{t} = 2t^{-1} \rightarrow y' = -2t^{-2} = \frac{-2}{t^2}$

to 1 less

Ex 6:

Find an equation which is tangent to $f(x) = 2x^2 - 3x$ @ $x = -1$

How to find a tangent line:

1) find $f'(x)$: $f'(x) = \lim_{h \to 0} \frac{2(x+h)^2 - 3(x+h) - (2x^2 - 3x)}{h}$

$\rightarrow \lim_{h \to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 \cancel{- 3x} - 3h \cancel{- 2x^2} \cancel{+ 3x}}{h}$

$\rightarrow \lim_{h \to 0} \frac{4xh + 2h^2 - 3h}{h} \rightarrow \lim_{h \to 0} \frac{\cancel{h}(4x + 2h - 3)}{\cancel{h}}$

$\rightarrow \lim_{h \to 0} 4x + 2h - 3 \rightarrow 4x + 2(0) - 3 \rightarrow \boxed{4x - 3 = f'(x)}$

2) find m @ $x = -1$ $f'(-1) = 4(-1) - 3 = -7 = m$ or $\frac{-7}{1}$

3) find $y_1$ $2(-1)^2 - 3(-1) \rightarrow 2 + 3 \rightarrow 5 = y_1$

4) Write equation in the form $y - y_1 = m(x - x_1)$

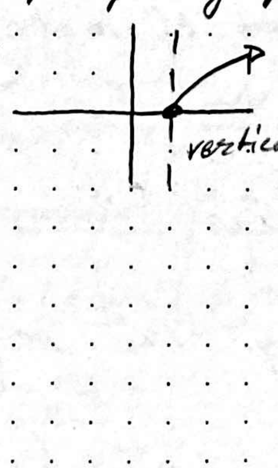
Slope form $\rightarrow y - 5 = -\frac{7}{1}(x + 1)$ or $y = -7(x+1) + 5$

sharp tip in graph = cusp $\rightarrow$ cannot be differentiated

$\rightarrow$ no defined slope

$\rightarrow$ uses vertical tangent line

Ex:



vertical tangent line, undefined slope

2.2 Warm-up

a) $g(x) = -3x^2 + x - 2 \rightarrow g'(x) = \lim_{h\to 0} \frac{-3(x+h)^2 + (x+h) - 2 - (-3x^2 + x - 2)}{h}$

$\rightarrow \lim_{h\to 0} \frac{\cancel{-3x^2} - 6xh - 3h^2 + \cancel{x} + h \cancel{-2} + \cancel{3x^2} - \cancel{x} + \cancel{2}}{h} \rightarrow \lim_{h\to 0} \frac{-6xh - 3h^2 + h}{h}$

$\rightarrow \lim_{h\to 0} \frac{h(-6x - 3h + 1)}{h} \rightarrow \lim_{h\to 0} -6x - 3h + 1 \rightarrow -6x + 1 = g'(x) \rightarrow$ plug in $x = 1$ $\hookrightarrow$ gives slope

b) $-3(1)^2 + 1 - 2 \rightarrow -4 \rightarrow (1, -4)$. $y + 4 = -5(x - 1)$ $y = -5(x-1) - 4$

2.2 Basic Differentiation Rules & Rates of change

The derivative of a constant function is 0
$\hookrightarrow$ as long as it's real

Ex 1

a) $y = 7 \rightarrow y' = 0$ b) $y = 0 \rightarrow y' = 0$

c) $y = -3 \rightarrow y' = 0$ d) $y = k\pi^2$, $k$ is constant $\rightarrow y'(x) = 0$

Power rule: If $n$ is a rational number, then the function $f(x) = x^n$ is differentiable and

$$\frac{d}{dx} x^n = n x^{n-1}$$

Ex 2

a) $y = x^3 \rightarrow y'(x) = 3x^2$ b) $g(x) = \sqrt[3]{x} \rightarrow x^{\frac{1}{3}} \rightarrow g'(x) = \frac{1}{3} x^{-2/3}$

c) $y = \frac{1}{x^2} \rightarrow x^{-2} \rightarrow y'(x) = -2x^{-3}$ or $\boxed{\frac{-2}{x^3}}$

Ex 3

$f(x) = x^2$ when $x = -2$ $(-2, 4)$ $f'(x) = 2x \rightarrow m = -4 \rightarrow$ tangent:
$y - 4 = -4(x + 2)$

Constant Multiple Rule

If $f$ is a differentiable function & $c$ is real, then $cf$ is also and $\frac{d}{dx}[cf(x)] = cf'(x)$

$$\rightarrow \boxed{\frac{d}{dx}[cx^n] = cnx^{n-1}}$$

Ex 4:

a) $y = 5x^3 \rightarrow \boxed{y' = 15x^2}$ b) $y = \frac{2}{x} \rightarrow 2x^{-1} \rightarrow y' = -2x^{-2}$ or $\boxed{\frac{-2}{x^2}}$

c) $f(t) = \frac{4t^2}{5} \rightarrow f'(x) = \frac{4}{5} \cdot 2t \rightarrow \boxed{\frac{8t}{5}}$

d) $y = 2\sqrt{x} \rightarrow 2x^{\frac{1}{2}} \rightarrow y' = 1x^{-\frac{1}{2}}$ or $\frac{1}{\sqrt{x}}$

HW: p. 114: 1, 4, 13, 23, 28, 35, 39, 41, 43, 49, 54, 59, ~~61~~ 60, 66, 67, 70, 79 (Pick 12)

e) $y = \frac{1}{2\sqrt[3]{x^2}} \to \frac{1}{2x^{\frac{2}{3}}} \to \frac{x^{-2/3}}{2}$   $f'(x) = \frac{1}{2}\cdot\left(-\frac{2}{3}\right)x^{-5/3}$

$\to -\frac{1}{3}x^{-5/3} \to -\frac{x^{-5/3}}{3} \to -\frac{1}{3\sqrt[3]{x^5}}$

f) $y = -\frac{5x}{2} \to -\frac{5}{2}\cdot 1x$

Ex 5:

a) $y = \frac{5}{2x^3} \to \frac{5}{2}\cdot\frac{x^{-3}}{1} \to \frac{5x^{-3}}{2}$   $\frac{dy}{dx} = \frac{5}{2}\cdot\frac{-3x^{-4}}{1}$

$\to \frac{5}{2}\cdot\frac{-3x^{-4}}{1} \to \frac{-15x^{-4}}{2} \to -\frac{15}{2x^4}$

b) $y = \frac{5}{(2x)^3} \to \frac{5}{8x^3} \to \frac{5}{8}\cdot\frac{x^{-3}}{1} \to \frac{5}{8}\cdot\frac{-3x^{-4}}{1} \to \frac{-15x^{-4}}{8}$

$\to \frac{-15}{8x^4}$

c) $y = \frac{7}{3x^{-2}} \to \frac{7}{3}\cdot\frac{x^2}{1} \to \frac{dy}{dx} = \frac{7}{3}\cdot\frac{2x}{1} \to \frac{14x}{3} \to \frac{14}{3x^{-1}}$

d) $y = \frac{7}{(3x)^{-2}} \to 7(3x)^2 \to 7(9x^2) \to 63x^2 \to \frac{d}{dx}$

Position Function, Velocity & Acceleration

- $s(t) = -16t^2 + v_0 t + s_0$
-
-

Sum & Difference Rules

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

Ex 6:

a) $f(x) = x^3 - 4x + 5$   $f'(x) = 3x^2 - \cancel{4x^0}1 + 0$   $3 - \frac{1}{x^2}$

$\to f'(x) = 3x^2 - 1$

b) $g(x) = -\frac{x^4}{2} + 3x^3 - 2x$   $g'(x) = -2x^3 + 9x^2 - 2$

c) $y = \frac{3x^2 - x + 1}{x} \to 3x - 1 + \frac{1}{x}$   $y' = 3\cancel{x^0} - 0 - x^{-2} \to 3 - x^{-2}$

Derivatives of Sine & Cosine

$\frac{d}{dx}[\sin x] = \cos x$   $\frac{d}{dx}[\cos x] = -\sin x$

Ex 7:

a) $y = 2\sin x \to y' = 2\cos x$   b) $y = \frac{\sin x}{2}$   $y' = \frac{\cos x}{2}$

c) $y = x + \cos x \to y' = 1x^0 + (-\sin x) \to y' = 1 - \sin x$

d) $\cos x - \frac{\pi}{3}\sin x$   $\frac{d}{dx} = -\sin x - \frac{\pi}{3}\cos x$

2.3 Product & Quotient Rules Sept. 19. 2024

Warm-up

1) $f(x) = 2\sqrt{x} - 4x^5 \Rightarrow 2x^{1/2} - 4x^5 \Rightarrow 1x^{-1/2} - 20x^4$

$\Rightarrow f'(x) = \frac{1}{\sqrt{x}} - 20x^4$

2) $y = \frac{3}{(3x)^3} \Rightarrow \frac{3x^{-3}}{27} \Rightarrow y' = \frac{3}{27} \cdot \frac{-3x^{-4}}{1} \Rightarrow \frac{-1}{3x^4}$ or $\frac{-x^{-4}}{3}$

Position, Velocity, & Acceleration

Position
Units: ft

$S(t) = -16t^2 + V_0 t + S_0$

initial velocity, initial height

Velocity
Units: $\frac{ft}{sec}$

$S'(t) = v(t) = -32t + V_0$

Acceleration
Units: $\frac{\left(\frac{ft}{sec}\right)}{sec}$ or $ft/sec^2$

$S''(t) = v'(t) = a(t) = -32$

The product Rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Ex 1: Derive

$h(x) = \underset{f(x)}{(3x - 2x^2)}\underset{g(x)}{(5+4x)} \Rightarrow h'(x) = (3x-2x^2)(4) + (5+4x)(3-4x)$

$\Rightarrow h'(x) = 12x - 8x^2 + 15 - 20x + 12x - 16x^2 \Rightarrow h'(x) = -24x^2 + 4x + 15$

Ex 2: Derive

$y = 3x^2 \sin x \Rightarrow y' = (3x^2)(\cos x) + (6x)(\sin x)$

Ex 3: Derive

$y = 2x\cos x - 2\sin x \Rightarrow y' = (2x)(-\sin x) + 2\cos x - 2\cos x \quad y' = -2x\sin x$

Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

high, low

$\frac{\text{low } d'\text{high} - \text{high } d'\text{low}}{\text{low}^2}$

Ex 4: Derive

$$y=\frac{5x-2}{x^2+1} \rightarrow y'=\frac{(x^2+1)(5)-(5x-2)(2x)}{(x^2+1)(x^2+1)} \rightarrow \frac{5x^2+5-10x^2+4x}{(x^2+1)^2}$$

$(x^2+1)(x^2+1)$ ← Don't foil!

$$\rightarrow y'=\frac{-5x^2+4x+5}{(x^2+1)^2}$$

Ex 5: Find tan. line

$$f(x)=\frac{3-\left(\frac{1}{x}\right)}{x+5} \quad @\ (-1,1) \rightarrow \frac{3-x^{-1}}{x+5} \rightarrow f'(x)=\frac{(x+5)(1x^{-2})-(3-x^{-1})(1)}{(x+5)^2}$$

$$\rightarrow f'(x)=\frac{\frac{x+5}{x^2}-\left(3-\frac{1}{x}\right)}{(x+5)^2}$$

$$m\ @ -1 \rightarrow f'(-1)=\frac{\frac{4}{1}-(4)}{(-1+5)^2} \rightarrow \frac{0}{16} \rightarrow m=0$$

tangent line equation! $y-1=0(x+1) \rightarrow \boxed{y=1}$ ✓

Ex 6: Derive

a) $y=\frac{x^2+3x}{6} \rightarrow \frac{1}{6}\cdot x^2+3x \rightarrow \frac{1}{6}\cdot 2x+3 \rightarrow \frac{2x+3}{6}=y'$

b) $y=\frac{5x^4}{8} \rightarrow \frac{5}{8}\cdot\frac{x^4}{1} \rightarrow \frac{5}{8}\cdot\frac{4x^3}{1} \rightarrow \frac{20x^3}{8} \rightarrow \frac{5x^3}{2}=y'$

c) $y=\frac{-3(3x-2x^2)}{7} \rightarrow -3\left(\frac{3x}{7x}-\frac{2x^2}{7x}\right)=-3\left(\frac{3}{7}-\frac{2x}{7}\right)$

d) $y=\frac{9x^{-2}}{5} \rightarrow y'=\frac{-18x^{-3}}{5} \rightarrow \frac{-18}{5x^3}$

Derivative of $\tan x$? $\rightarrow \frac{\sin x}{\cos x} \rightarrow y'=\frac{\cos x(\cos x)-\sin x(-\sin x)}{(\cos x)^2}$

$$\rightarrow \frac{\cos^2 x+\sin^2 x}{(\cos x)^2} \rightarrow \frac{1}{\cos^2 x}=\sec^2 x$$

Derivative of $\cot x$? $\rightarrow \frac{\cos x}{\sin x} \rightarrow -\csc^2 x \qquad y'=\frac{\sin x(-\sin x)-\cos x(\cos x)}{\sin^2 x}$

$$\rightarrow \frac{-\sin^2 x-\cos^2 x}{\sin^2 x} \rightarrow \frac{-1(\sin^2 x+\cos^2 x)}{\sin^2 x} \rightarrow \frac{-1}{\sin^2 x} \rightarrow -\csc^2 x$$

Derivative of Trig. functions

$\frac{d}{dx}\sec x=\sec x\tan x \qquad \frac{d}{dx}\csc x=-\csc x\cot x$

$\frac{d}{dx}\cos x=-\sin x \qquad \frac{d}{dx}\sin x=\cos x$

$\frac{d}{dx}\tan x=\sec^2 x \qquad \frac{d}{dx}\cot x=-\csc^2 x$

Ex 7: Derive

a) $y=x-\tan x \rightarrow y'=1-\sec^2 x=-\tan^2 x$

b) $y=x|\sec x \rightarrow y'=x(\sec x\tan x)+\sec x$

$y'=x\sec x\tan x+\sec x$

Ex. 8: Differentiate & Prove both sides

$$y = \frac{1-\cos x}{\sin x} = \csc x - \cot x$$

$$\frac{dy}{dx} = \frac{\sin x(\sin x) - (1-\cos x)(\cos x)}{\sin^2 x} = -\csc x \cot x + \csc^2 x$$

$$\frac{\sin^2 x - \cos x + \cos^2 x}{\sin^2 x}$$

$$\frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x}$$

$$\csc^2 x - \frac{\cos x}{\sin x} \cdot \frac{1}{\sin x}$$

$$\csc^2 x - \cot x \csc x = -\csc x \cot x + \csc^2 x$$

HW: p. 125 Q: 1, 5, 11, 15, 19, 25, 30, 37, 41, 50, 60, 65a, 69, 79

2.4 Chain Rule

Warm up $f(x)=\sqrt{x+2}$ & $g(x)=x^2+4$ find $f\circ g$ & $g\circ f$

$\sqrt{(x^2+4)+2}=f\circ g$

$\rightarrow \sqrt{x^2+6}$

$(\sqrt{x+2})^2+4=g\circ f$

$x+2+4 \rightarrow x+6$

$h(x)=\sqrt{x}$ $k(x)=x^2+6$

$h(k(x))=\sqrt{x^2+6}$

Without the Chain Rule

$y=x^2+1$
$y=\sin x$
$y=3x+2$
$y=x+\tan x$

With the Chain Rule

$y=\sqrt{x^2+1}$
$y=\sin 6x$
$y=(3x+2)^5$
$y=x+\tan x^2 \neq \tan^2 x$

The Chain Rule

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}[f(g(x))]=f'(g(x))g'(x)$$

Ex 1.

a) $y=\frac{1}{x+1}$ $u=x+1$ $y=\frac{1}{u}$

b) $y=\sin 2x$ $u=2x$ $y=\sin u$

c) $y=\sqrt{3x^2-x+1}$ $u=3x^2-x+1$ $y=\sqrt{u}$ or $u^{1/2}$

d) $y=\tan^2 x$ $=(\tan x)^2$ $u=\tan x$ $y=u^2$

Ex. 2

$y=(x^2+1)^3$ $u=x^2+1$ $y=u^3$

$\frac{dy}{dx}=3(x^2+1)^2(2x)=\boxed{6x(x^2+1)^2}$

General Power Rule

$$\frac{dy}{dx}=n[u(x)]^{n-1}\frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}[u^n]=nu^{n-1}u'$$

Ex 3 $f(x)=(3x-2x^2)^3$ $u=3x-2x^2$ $y=u^3$

$f'=3(3x-2x^2)^2(3-4x) \rightarrow \boxed{9-12x(3x-2x^2)^2}$

Ex 4 $f(x)=\sqrt[3]{(x^2-1)^2}$ where $f'(x)=0$ & where $f'(x)$ dne

$f(x)=(x^2-1)^{2/3}$ $u=x^2-1$ $y=u^{2/3}$ $f'(x)=\frac{2}{3}(x^2-1)^{-1/3}(2x)$

$\rightarrow f'(x)=\frac{4}{3}x(x^2-1)^{-1/3}$ or $\frac{4x}{3\sqrt[3]{x^2-1}}$

$f'(x) \rightarrow 0 = \frac{4x}{3\sqrt[3]{x^2-1}} \rightarrow 0 = 4x \rightarrow 0 = x$

(0, 1) original $f(x) \rightarrow$ eval.: $\sqrt[3]{(0^2-1)^2} = 1$ } Finding where $f'(x) = 0$

↳ Where $f'(x)$ is 0

$f'(x)$ does not exist when the denominator is 0

$f'(x)$ denominator $\rightarrow 3\sqrt[3]{x^2-1} = 0 \rightarrow \sqrt[3]{x^2-1} = 0$

$\rightarrow x^2 - 1 = 0 \rightarrow x^2 = 1 \rightarrow x = \pm 1$

(1, 0) (-1, 0)

Points where $f'(x)$ dne

$f(1) = \sqrt[3]{(1^2-1)^2} = 0$

$f(-1) = \sqrt[3]{(-1^2-1)^2} = 0$

HW: P. 136 Q: 1, 5, 10, 11, 13, 21, ~~27~~, ~~31~~, 45, 59, 67, ~~77~~, (81-89) odd

2.4 Day 2 Sept. 26 2024

Ex 5:

$g(t) = \frac{-7}{(2t-3)^2}$ $u = 2t-3$ $y = \frac{-7}{u^2} \rightarrow y = -7u^{-2}$

$\frac{d}{dx} = 14(2t-3)^{-3}(2) \rightarrow 28(2t-3)^{-3} \rightarrow \frac{28}{(2t-3)^3}$

Ex 6: Product & Chain Rule

$f(x) = x^2\sqrt{1-x^2} \rightarrow x^2(1-x^2)^{1/2} \rightarrow u = 1-x^2 \quad y = u^{1/2}$

$f'(x) = x^2\left(\frac{1}{2}(1-x^2)^{-1/2}(-2x)\right) + 2x(1-x^2)^{1/2}$

$= \frac{-x^3}{\sqrt{1-x^2}} + 2x\sqrt{1-x^2}$

Ex 7:

$f(x) = \frac{x}{\sqrt[3]{x^2+4}} \rightarrow x(x^2+4)^{-1/3} \rightarrow u = x^2+4 \quad y = u^{-1/3}$

$f'(x) = x\left(-\frac{1}{3}(x^2+4)^{-4/3}(2x)\right) + 1\left((x^2+4)^{-1/3}\right) = -\frac{2x^2}{3(x^2+4)^{4/3}} + \frac{1}{(x^2+4)^{1/3}}$

Ex 8:

$y = \left(\frac{3x-1}{x^2+3}\right)^2 \quad y' = 2\left(\frac{3x-1}{x^2+3}\right)^1\left(\frac{(x^2+3)(3)-(3x-1)(2x)}{(x^2+3)^2}\right)$

$= \frac{(6x-2)(3x^2+9-6x^2+2x)}{(x^2+3)^3}$

Trig funcs with chain rule

$\frac{d}{dx}[\sin u] = (\cos u)u'$ $\frac{d}{dx}[\cos u] = -(\sin u)u'$

$\frac{d}{dx}[\tan u] = (\sec^2 u)u'$ $\frac{d}{dx}[\cot u] = -(\csc^2 u)u'$

$\frac{d}{dx}[\sec u] = (\sec u \tan u)u'$ $\frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$

Ex 9

a) $y = \sin 2x \rightarrow y' = (\cos(2x))(2) \rightarrow 2\cos 2x$

b) $y = \cos(x-1)$ $y' = -\sin(x-1)(1) \rightarrow -\sin(x-1)$

c) $y = \tan 3x$ $y' = (\sec^2(3x))(3) \rightarrow 3\sec^2(3x)$

Ex 11:

$f(t) = \sin^3 4t \rightarrow (\sin(4t))^3$ $u = \sin 4t$ $y = u^3$

$\rightarrow y' = 3(\sin 4t)^2 (\cos(4t)(4)) \rightarrow 3(\sin 4t)^2 4\cos(4t) \rightarrow 12\sin^2(4t)\cos(4t)$

Ex. 11.5:

$f(x) = \sec^4(3x) \rightarrow (\sec 3x)^4$ $u = \sec 3x$ $y = u^4$

$u' = \sec 3x \tan 3x(3) = 3(\sec 3x \tan 3x)$

$f'(x) = 4(\sec 3x)^3(3(\sec 3x \tan 3x)$

$\rightarrow 12\sec^3 3x(\sec 3x \tan 3x)$

HW: p 136 (1, 8, 10, 11, 13, 21, 27, 31, 45, 59, 67, 77, 81-89 odd)

# 2.5 Implicit Differentiation

Oct. 1. 2024

Warm up → 1) $f(x) = \sqrt[3]{2x-1}$ → $(2x-1)^{1/3}$ @ $x=-1$ find tangent:

→ $f'(x) = \frac{1}{3}(2x-1)^{-2/3}(2)$ → $\frac{7}{3}(2x-1)^{-2/3}$ → $\frac{7}{3\sqrt[3]{(2x-1)^2}} = f'(x)$

→ $\frac{2}{3}(2x-1)^{-2/3}$ → $\frac{2}{3(2x-1)^{2/3}}$ → $f'(-1) = \frac{2}{3(2(-1)-1)^{2/3}}$

$m = 0.32$ @ $x=-1$, $y = -1.442$

→ $y + 1.442 = 0.32(x+1)$

2) $g(x) = \sec 3x + \cot^2 x$ $g'(x) = 3\sec 3x \tan 3x - 2\cot x \csc^2 x$

---

2.5 Implicit Equation: Equation where you cannot solve for $y$

Functions in implicit form:

$xy^2 + y^3 = x + 4$ → can't solve for y.

Ex: Find what y equals

$x^2 + y^2 = 25$

Unnecessarily complicated

$\sqrt{y^2} = \sqrt{25 - x^2}$ → $y = \pm\sqrt{25-x^2}$

When deriving implicitly, differentiation is taking place with respect to $x$

$\frac{d}{dx}(x^4) = 4x^3$

$\frac{d}{dx} \sin x = \cos x$

$\frac{d}{dx}(y) = \frac{dy}{dx}$

$\frac{d}{dx}(y^2) = 2y^1\left(\frac{dy}{dx}\right)$

$\frac{d}{dx}(y^8) = 8y^7\left(\frac{dy}{dx}\right)$

$\frac{d}{dx}(\sin y) = \cos y\left(\frac{dy}{dx}\right)$

$\frac{d}{dx}(z) = \frac{dz}{dx}$

$\frac{d}{dx}(z^3) = 3z^2\left(\frac{dz}{dx}\right)$

$\frac{d}{dx}(\sec z) = \sec z \tan z\left(\frac{dz}{dx}\right)$

Ex 1:

a) $\frac{d}{dx}(x^4) = 4x^3$

b) $\frac{d}{dx}(y^4) = 4y^3\left(\frac{dy}{dx}\right)$

c) $\frac{d}{dx}(x+3y) = \left(1 + 3\left(\frac{dy}{dx}\right)\right)$

d) $\frac{d}{dx}(xy^2) = x2y\frac{dy}{dx} + 1y^2$

→ $2xy\frac{dy}{dx} + y^2$

Guidelines:

1) Differentiate both sides of the equation with respect to $x$

2) Collect all terms involving $\frac{dy}{dx}$ on the left side of the equation & move all other terms to the right side of the equation

3) Factor $\frac{dy}{dx}$ out of the left side of the equation

4) Solve for $\frac{dy}{dx}$

Ex 2: Find $\frac{dy}{dx}$ given:

$y^3+y^2-5y-x^2=-4 \rightarrow \frac{d}{dx}(y^3+y^2-5y-x^2)=\frac{d}{dx}(-4)$

$\rightarrow 3y^2\left(\frac{dy}{dx}\right)+2y\left(\frac{dy}{dx}\right)-5\left(\frac{dy}{dx}\right)-2x=0$

$\rightarrow 3y^2\left(\frac{dy}{dx}\right)+2y\left(\frac{dy}{dx}\right)-5\left(\frac{dy}{dx}\right)=2x$

$\rightarrow \frac{dy}{dx}(3y^2+2y-5)=2x \rightarrow \frac{dy}{dx}=\left(\frac{2x}{3y^2+2y-5}\right)$

Ex 3:

a) $x^2+y^2=0 \rightarrow \sqrt{y^2}=\sqrt{-x^2} \rightarrow$ non-real
ignore for calc

b) $x^2+y^2=1 \rightarrow y=\pm\sqrt{1-x^2}$
differentiable

c) $x+y^2=1 \rightarrow y=\pm\sqrt{1-x}$
differentiable

Ex 4: determine slope @ $(\sqrt{2}, -1/\sqrt{2})$ of

$x^2+4y^2=4 \rightarrow \frac{d}{dx}(x^2+4y^2)=\frac{d}{dx}(4) \rightarrow 2x+8y\left(\frac{dy}{dx}\right)=0$

$\rightarrow \frac{dy}{dx}(8y)=-2x \rightarrow \frac{dy}{dx}=-\frac{2x}{8y} \rightarrow -\frac{x}{4y}$

$m @ \left(\sqrt{2}, -\frac{1}{\sqrt{2}}\right) = \frac{dy}{dx}\Big|_{\left(\sqrt{2}, -\frac{1}{\sqrt{2}}\right)} = \frac{-\sqrt{2}}{-4/\sqrt{2}} \rightarrow \frac{-\sqrt{2}}{1}\cdot\frac{\sqrt{2}}{-4} = \frac{-2}{-4}$

$\rightarrow \frac{1}{2}=m$

Ex 5: Derive implicitly.

$xy^3+y^2+2x=-10 \rightarrow \frac{d}{dx}(xy^3+y^2+2x)=\frac{d}{dx}(-10)$

$\rightarrow \left(x\left(3y^2\left(\frac{dy}{dx}\right)\right)+1(y^3)\right)+2y\left(\frac{dy}{dx}\right)+2=0$

$3xy^2\left(\frac{dy}{dx}\right)+y^3+2y\left(\frac{dy}{dx}\right)+2=0 \rightarrow \frac{dy}{dx}(3xy^2+2y)=-y^3-2$

$\rightarrow \frac{dy}{dx}=\frac{-2-y^3}{(3xy^2+2y)}$

Ex 7: Second derivative implicitly ☆ On quiz & test! ☆

$x^2+y^2=25$ find $\frac{d^2y}{dx^2}$

$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(25) \to 2x+2y\left(\frac{dy}{dx}\right)=0$

$\to \left(\frac{dy}{dx}\right)(2y)=-2x \to \frac{dy}{dx}=\frac{-2x}{2y} \to \frac{-x}{y}$ first derivative

$\frac{d^2y}{dx^2}=\frac{y(-1)-(-x)\left(\frac{dy}{dx}\right)}{y^2} = \frac{\frac{y}{y}\cdot -y + x\left(\frac{-x}{y}\right)}{y^2}$

$\to \frac{\cancel{\frac{-y^2-x^2}{y}}}{y^2} \to \frac{\frac{(-1)y^2+x^2}{y}}{y^2} = \frac{\frac{(-1)25}{y}}{y^2} \to \frac{-25}{y^3}$ (check original equation)

HW: 145 (1, 5, 9, 11, 21, 27, 29, 34, 37, 60, 68)

2-6

WU: $x^3y^3 - y = x \rightarrow \left(x^3\left(3y^2\frac{dy}{dx}\right) + 3x^2(y^3)\right) - \frac{dy}{dx} = 1$

$\rightarrow \frac{dy}{dx}(x^3 3y^2 - 1) = 1 - 3x^2y^3$

$\rightarrow \boxed{\frac{dy}{dx} = \frac{1-3x^2y^3}{(x^3 3y^2 - 1)}}$

@ $(0,0)$

$\rightarrow \frac{1-3(0)^2(0)^3}{0^3 3(0)^2 - 1} \rightarrow \frac{1}{-1} \rightarrow -1$

$\left.\frac{dy}{dx}\right|_{(0,0)} = -\frac{1}{1}$

Differentiating with respect to "t" using implicit differentiation

$\frac{d}{dt}(x) = \frac{dx}{dt}$ $\quad \frac{d}{dt}(x^2) \rightarrow 2x\frac{dx}{dt}$ $\quad \frac{d}{dt}(x^5) \rightarrow 5x^4\frac{dx}{dt}$

$\frac{d}{dt}(y) = \frac{dy}{dt}$ $\quad \frac{d}{dt}(y^3) = 3y^2\frac{dy}{dt}$ $\quad \frac{d}{dt}(z) = \frac{dz}{dt}$ $\quad \frac{d}{dt}(V) = \frac{dV}{dt}$

$\frac{d}{dt}(r^2) = 2r\frac{dr}{dt}$

## 2.6 Related Rates

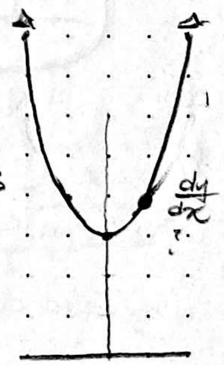
Related Rates - slopes of 2 or more related variables that are changing with respect to time

Ex - Pouring water

Constants: Size of containers (Radius), Color, amount of water existing

Variable (Things that change): Liquid in 2nd container, Air decreased in 2nd, container, flow rate, "shape of water"

Related Rates: $\frac{dV}{dt}$ $\quad \frac{dh}{dt}$



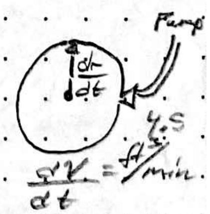

Ex 1: $x$ & $y$ are diff. func. of $t$. & are related by $y = x^2+3$
Find $\frac{dy}{dt}$ when $x=1$ given $\frac{dx}{dt} = 2$ when $x=1$

Given $x=1$ $\quad$ Find $\frac{dy}{dt}$

$\frac{dx}{dt} = 2$ $\quad y = x^2+3$

$\frac{d}{dt} \rightarrow \frac{dy}{dt} = 2x\frac{dx}{dt} = 2(1)(2)\ \boxed{=4}$

Ex 2: Air into sphereical balloon @ rate of 4.5 cubic ft. per min
Find rate of change of the radius when radius is 2 feet




Given: $\frac{dV}{dt} = {}^+4.5\ ft^3/min$ $\quad r = 2\ ft$

find $\frac{dr}{dt}$ $\quad$ Use: $V = \frac{4}{3}\pi r^3$ $\quad \frac{dV}{dt} = \frac{4\pi}{3}\left(3r^2\frac{dr}{dt}\right) \rightarrow 4\pi r^2\frac{dr}{dt}$

$4.5 = 4\pi(2)^2\frac{dr}{dt} \rightarrow \frac{4.5}{16\pi} = \frac{dr}{dt}$

$\rightarrow 0.9 = \frac{dr}{dt} \rightarrow$ The radius is changing @ 0.9 ft/min when $r = 2$

Unit 2 Review

① $f(x)=12 \quad \frac{dy}{dx}=\lim_{h\to0}\frac{f(x+h)-f(x)}{h} \to \frac{-12}{h} \to 0$

③ $f(x)=x^2-4x+5 \to \lim_{h\to0}\frac{(x+h)^2-4(x+h)+5-(x^2-4x+5)}{h}$

$\to \frac{\cancel{x^2}+2xh+h^2\cancel{-4x}-4h\cancel{+5}\cancel{-x^2}\cancel{+4x}\cancel{-5}}{h} \to \frac{2xh+h^2-4h}{h} \to \frac{h(2x+h-4)}{h}$

$\to \lim_{h\to0} 2x+h-4 \to 2x-4$

Warm up: Find $\frac{d^2y}{dx^2}$ of $4x^2+2y^2=8$ $\quad \frac{-2x}{y}$

$\to 8x+4y\frac{dy}{dx}=0 \to \frac{dy}{dx}=\frac{-8x}{4y} \to \frac{d^2y}{dx^2}=$

$\to \frac{y(-2)+(2x)\left(\frac{dy}{dx}\right)}{y^2} \to \frac{-2y+2x\frac{dy}{dx}}{y^2} \to \frac{-2y+2x\left(\frac{-2x}{y}\right)}{y^2}$

$$\frac{\frac{-2y}{1}\cdot\frac{y}{y}+2x\left(\frac{-2x}{y}\right)}{y^2} \to \frac{\frac{-2y^2}{y}+\frac{-4x^2}{y}}{y^2} \to \frac{-2y^2-4x^2}{y}\cdot\frac{1}{y^2}$$

$$=\frac{-1(2y^2+4x^2)}{y^3}=\frac{d^2y}{dx^2}$$

Practice trig derivatives!

d/dx sinx = cosx  d/dx cosx = -sinx  d/dx tanx = $\sec^2x$

d/dx cotx = $-\csc^2x$  d/dx secx = secxtanx  d/dx csc = -cscxcotx

d/dx cosx = -sinx  d/dx sinx = cosx  d/dx sec = secxtanx

d/dx tanx = $\sec^2x$  d/dx cot x = $-\csc^2x$  d/dx cscx = -cscxcotx

$\cos^2x+\sin^2x=1$  $1+\tan^2x=\sec^2x$  $1+\cot^2x=\csc^2x$

sin = cosx  cosx = -sinx  secx = secxtanx

cscx = -cscxcotx  tanx = $\sec^2x$  cot = $-\csc^2x$

d/dx sin = cosx  d/dx cosx = -sinx  d/dx secx = secxtanx

d/dx cscx = -cscxcotx  $\frac{d}{dx}$tanx = $\sec^2x$  $\frac{d}{dx}$ cotx = $-\csc^2x$

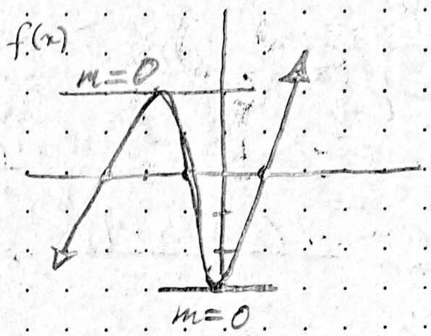
pyth

$\sin^2x+\cos^2x=1$  $1+\tan^2x=\sec^2x$  $1+\cot^2x=\csc^2x$

3.1 Extrema on an Interval Oct 22 2024

Warm-up: relative extrema of $f(x)=(x+3)(x^2-1) \to x^3-x+3x^2-3=f(x)$

Maximum: $x=-2.155$ Minimum: $x=0.155$
$y=3.079$ $y=-3.079$




There is a relative maximum of 3.079 at $x \approx -2.155$.
There is a relative minimum of $-3.079$ at $x \approx 0.155$.

$m=f'(x)= (x+3)(2x)+(x^2-1)(1)$

x-vals: max: $-2.155$ min: $0.155$

$(-\infty, -2.155)$ $(-2.155, 0.155)$ $(0.155, \infty)$ $\quad {}^{L} f'(x)=(x+3)(2x)+(x^2-1)$

f' $\quad$ −3 (+) $\quad$ 0 (−) $\quad$ 2 (+)

f

3.1

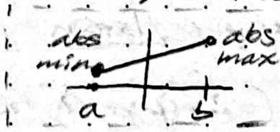
find relative & absolute extrema

Definition of Extrema → Let $f$ be on interval $I$, containing c.

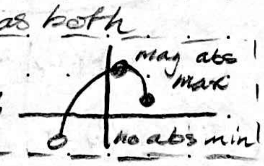
1. $f(c)$ is the minimum of $f$ on $I$ when $f(c) \leq f(x)$ for all $x$ on $I$
2. $f(c)$ is the maximum of $f$ on $I$ when $f(c) \geq f(x)$ for all $x$ on $I$

Extreme Value Theorem Absolute max & min guaranteed on closed interval

If $f$ is continuous on a closed interval $[a, b]$ then $f$ has both minimum & maximum on interval




$[a, b]$:




Definition of Relative Extrema

1: Open interval containing "c" on which $f(c)$ is a maximum, then $f(c)$ is called a relative maximum of "f" or $f$ has relative maximum @ $(c, f(c))$.

2. " is a minimum, "
relative minimum " relative minimum @ $(c, f(c))$.

Relative = local Absolute = global

Definition of Critical Number

If $f'(c)=0$ or "f" doesn't have $f'(c)$, then c is a critical num. of "f"

Critical numbers are potential mins. or maxs. of "f"

Include vert. asymptotes, x-vals. of holes on critical number list

Relative Extrama Occur Only @ critical numbers

Vertical asymptotes & holes aren't Extrema

Ex 2: Find Extrema of $f(x) = 2\sin x - \cos 2x$ on $[0, 2\pi]$ abs plugin end points

find $f'$ → $f'(x) = 2\cos x + \sin 2x(2)$ → $2\cos x + 2\sin 2x = f'(x)$

$\sin 2x = 2\cos x + 2(2\sin x \cos x)$

" $= 2\cos x(1 + 2\sin x)$

" $= 2\cos x \quad 1 + 2\sin x = 0$

$\sin x = -\frac{1}{2}$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

$x = \frac{7\pi}{6}, \frac{11\pi}{6}$

$f(x) = 2\sin x - \cos 2x$

$x = \frac{\pi}{2}$ $2(1) - (-1) \to 3$

$x = \frac{3\pi}{2}$ $2(-1) - (-1) \to -1$

$x = \frac{7\pi}{6}$ $2(-\frac{1}{2}) - (-\frac{1}{2}) \to -1.5$

$x = \frac{11\pi}{6}$ $2(-\frac{1}{2}) - (\frac{1}{2}) \to -1.5$

$x = 2\pi$ $(2(0)) - (1)) \to -1$

Plug into original

| $x$ | 0 | $\pi/2$ | $\frac{3\pi}{2}$ | $\frac{7\pi}{6}$ | $\frac{11\pi}{6}$ | $2\pi$ |
|---|---|---|---|---|---|---|
| $f(x)$ | 0 | 3 | -1 | -1.5 | -1.5 | -1 |

abs max = 3 @ $x = \pi/2$

abs min $x = -3/2$ @ $x = \frac{7\pi}{6}, \frac{11\pi}{6}$

How to find absolute Extrema on [a, b]

1. find $f'(x)$ 2. Set $f'(x)$ = to "0" & find critical values

3. Plug in all critical values & end points into $f(x)$

4. Interpret results ⇒ highest y-value is the abs max & lowest is abs min

How to find relative Extrema

1. find $f'(x)$ 2. Set $f'(x)$ equal to "0" & find critical values

3. Make intervals 4. Select test point from every interval & plug it into $f'(x)$ - derivative

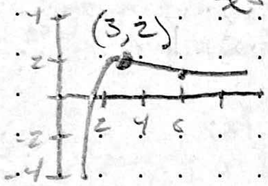
5. If there is a sign change @ critical value, that c.v. is a location of a relative extrema

Change from + to − is rel. max.
from − to + is rel. min.

no change = no extrema

Ex 1: Derivative @ Rel. Extrema

ⓐ $f(x)=\dfrac{9(x^2-3)}{x^3}$ $f'(x)=\dfrac{x^3(9(x^2-3)(2x))-(9(x^2-3))(3x^2)}{x^6}$

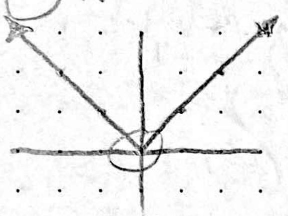



$f'(3)=0$

$(m @ x=3)=0$

maximum of 2 @ $x=3$.

ⓑ $f(x)=|x|$

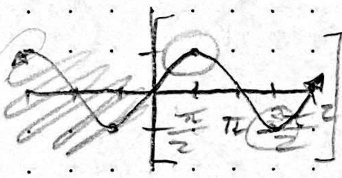


minimum of "0" @ $x=0$

no slope → cusp/sharp curve

not differentiable @ $x=0$

ⓒ $f(x)=\sin x$ on $[0,2\pi]$ $f'(x)=\cos x$




max of 1 @ $x=\frac{\pi}{2}$ $f'(x)=0$ @ $x=\frac{\pi}{2},\frac{3\pi}{2}$

min of $-1$ @ $x=\frac{3\pi}{2}$

3.3 Increasing & Decreasing Functions & 1st $\frac{dy}{dx}$ test (Oct. 24. 2024)

Definition of Increasing & Decreasing Functions:

A function is increasing on the interval $x_1$ & $x_2$ if $x_1<x_2$ implies $f(x_1)<f(x_2)$.

A function is decreasing on $x_1$ & $x_2$ if $x_1<x_2$, implies $f(x_1)>f(x_2)$

Test for increasing/decreasing functions:

if on interval $[a,b]$ continuous & differentiable on $(a,b)$;

1: if $f'(x)>0$ for all $x$ on $(a,b)$, $f$ is increasing on $[a,b]$

2: if $f'(x)<0$ for all $x$ on $(a,b)$, $f$ is decreasing on $[a,b]$

3: if $f'(x)=0$ for all $x$ on $(a,b)$, $f$ is constant on $[a,b]$

Ex 1:

$f(x)=x^3-4x$ $f'(x)=3x^2-4$ $0=3x^2-4 \to 4=3x^2$

$\to 4/3=x^2 \to x=\pm\sqrt{\frac{4}{3}} \to x=\pm\frac{2}{\sqrt{3}}$

$(-\infty,-\frac{2}{\sqrt{3}})(-\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}})(\frac{2}{\sqrt{3}},\infty)$ | $f(x)$ is increasing on $(-\infty,-\frac{2}{\sqrt{3}})\cup(-\frac{2}{\sqrt{3}},\infty)$.

$-3$ $0$ $3$

$f'$ $+$ $-$ $+$ | $f(x)$ is decreasing on $(-\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}})$

Ex 2: $f(x)=x^3-\frac{3}{2}x^2$ $\quad f'(x)=3x^2-3x$

$0=3x^2-3x \to 0=3x(x-1) \to x=0,1$

$(-\infty,0)(0,1)(1,\infty)$

$\quad -1 \qquad \frac{1}{2} \qquad 2$

$f'$: $+ \qquad - \qquad +$

f(x) is increasing on $(-\infty,0)\cup(1,\infty)$
f(x) is decreasing on $(0,1)$

Monotonic functions - No mins/maxs., only increasing/decreasing

Ex: $(\cdot,\cdot)(\cdot,\cdot)$ with $+ \quad +$ or $(\cdot,\cdot)(\cdot,\cdot)$ with $- \quad -$

Ex 3: $f(x)=x^3-12x-5$ $\quad f'(x)=3x^2-12$

$0=3x^2-12 \to 12=3x^2 \to 4=x^2 \to x=\pm 2$

$(-\infty,-2)(-2,2)(2,\infty)$

$\quad -3 \qquad 0 \qquad 3$

$f'$: $+ \qquad - \qquad +$

Local (Relative) max is 11 @ $x=-2$.
Local (Relative) min. is $-21$ @ $x=2$.

Use derivative! $\quad -2^3-12(-2)-5 \quad -8+24-5 \to 11$

$2^3-12(2)-5 \to 8-24-5 \to -21$

first derivative test!

if slope changes from + to −, or vice versa, there is a rel max or min.

# 3.2 Rolle's Theorem & Mean Value Theorem — Oct. 25, 2024

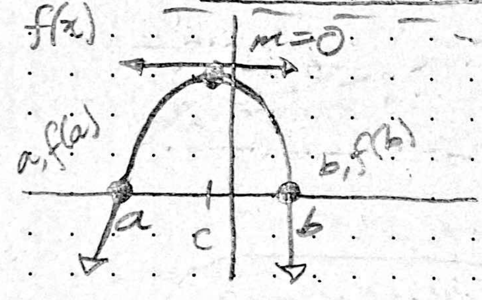
## Rolle's Theorem

f is continuous on [a, b] & differentiable on (a, b)

if $f(a) = f(b)$, ~~then at least 1~~ there is number "c" in (a, b)

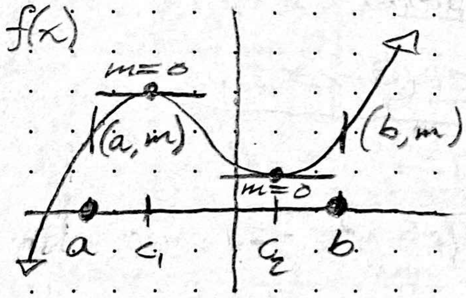
such that $f'(c) = 0$

Ex




3 conditions:
- f(x) must be continuous on [a, b]
- f(x) must be differentiable on (a, b)
- f(a) = f(b) (y-vals are equal)

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{0}{n} = 0 \rightarrow \text{(for Rolle's Theorem)}$$




**Ex 1:** $f(x) = x^2 - 3x + 2$ ✓✓✓

$0 = x^2 - 3x + 2 \rightarrow 0 = (x-2)(x-1)$

$x = 1; 2$ $\quad f'(x) = 2x - 3$

$0 = 2x - 3 \rightarrow 3 = 2x \rightarrow x = 3/2 = c$

**Ex 2:**

$f(x) = x^4 - 2x^2$ $\quad f'(x) = 4x^3 - 4x$ $\quad$ on (-2, 2)

$f(-2) = (-2)^4 - 2(-2)^2 \rightarrow 16 - 8 \rightarrow 8$ $\qquad f(2) = (2)^4 - 2(2)^2 \rightarrow 16 - 8 \rightarrow 8$

differentiable ✓ $\quad$ continuous ✓ $\quad$ $f(a) = f(b)$ ✓

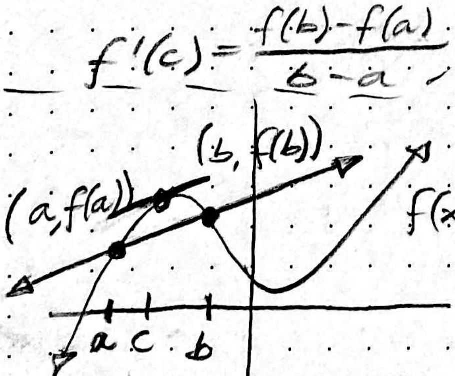
$0 = 4x^3 - 4x \rightarrow 0 = 4x(x^2 - 1) \rightarrow 0 = x \rightarrow 0 = x^2 - 1 \rightarrow x = \pm 1$

$c = 0, \pm 1$

## Mean Value Theorem:

If f is continuous on [a, b] & differentiable on (a, b), then there exists a "c" in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$




**Ex 3:** $f(x) = 5 - \frac{4}{x}$ on $(1, 4)$ $\quad$ a b

$f(1) = 5 - 4 \rightarrow 1 \rightarrow (1, 1)$ $\quad f(4) = 5 - 1 \rightarrow 4 \rightarrow (4, 4)$

continuous ✓

$f'(x) = \frac{-x(0) + 4(1)}{x^2} \rightarrow \boxed{\frac{4}{x^2}}$ $\qquad \frac{f(4) - f(1)}{4 - 1} = \frac{4}{x^2}$

$\frac{4-1}{4-1} = \frac{4}{x^2} \rightarrow 1 = \frac{4}{x^2} \rightarrow x = \pm 2$ ★ $\quad c = 2$ ★

## 3.4 Concavity & 2nd derivative test

Oct. 29. 2024

WU: $f(x) = \frac{x-1}{x^2} \rightarrow f'(x) = \frac{x^2(1) - (x-1)(2x)}{x^4} \rightarrow \frac{x^2 - 2x^2 + 2x}{x^4} \rightarrow \frac{-x^2+2x}{x^4}$

$0 = \frac{-x^2+2x}{x^4} \rightarrow 0 = -x^2 + 2x \rightarrow 0 = x(-x+2) \rightarrow x = 0 = c$

Asymptote

$0 = -x + 2 \rightarrow x = 2$

$(-\infty, 0)(0, 2)(2, \infty)$

$-1 \quad 1 \quad 3$

$f'(x)$: $-$ $+$ $-$

$\frac{-(-1)^2 + 2(-1)}{(-1)^4} \rightarrow -1 - 2 \rightarrow -3$

$\frac{-(1)^2 + 2(1)}{(1)^4} \rightarrow \frac{-1+2}{1} \rightarrow 1$

$\frac{-(3)^2 + 2(3)}{3^4}$

$-9 + 6$

$f(x)$ increases on $(0, 2)$ $f(x)$ decreases on $(-\infty, 0) \cup (2, \infty)$

~~f(0) = (0-1)/0^2 undefined~~ | $f(2) = \frac{2-1}{4} \rightarrow \frac{1}{4} \rightarrow (2, 1/4)$ relative maximum

---

**Concavity** $\rightarrow$ Hollowed / Curved Inward

concave up: "cup" | concave down: "frown"

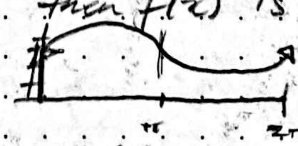
**Definition:** Let $f$ be differentiable on $I$. Graph of $f$ is concave up on $I$ when $f'$ is increasing & $f$ is concave down when $f'$ is decreasing.

**Theorem 3.7** Test for concavity

If $f''(x) > 0$ for all $I$, then $f(x)$ is concave upward.

If $f''(x) < 0$ for all $x$ in $I$, then $f(x)$ is concave downward.

**Ex 1:** $y = 3 + \sin(x)$ on $(0, 2\pi)$




(a)

$y' = \cos(x) \quad y'' = -\sin(x) \quad 0 = -\sin(x) \quad x = \pi$

$(0, \pi)(\pi, 2\pi)$ $f(x)$ is concave up on $(\pi, 2\pi)$.

$\pi/2 \quad 3\pi/2$

$y''$: $-$ $+$ $f(x)$ is concave down on $(0, \pi)$.

concave down, concave up

(b) $y = 4x^3 + 21x^2 + 36x - 20 \rightarrow y' = 12x^2 + 42x + 36$ $y'' = 24x + 42$

$0 = 24x + 42 \rightarrow -42 = 24x \rightarrow x = \frac{-42}{24} \rightarrow x = -\frac{7}{4}$

$(-\infty, -\frac{7}{4})(-\frac{7}{4}, \infty)$

$-2 \quad 0$

$f''$: $-$ $+$

concave down, concave up

$24(-2) + 42 \rightarrow -48 + 42 \rightarrow -6$

$24(0) + 42 \rightarrow 42$

$f(x)$ is concave up on $(-7/4, \infty)$ & concave down on $(-\infty, -7/4)$.

Point of inflection: ~~Vertical tangent line~~ Point of function where concavity switches

→ Point! Full coordinate required! → $(x, y)$

Theorem 3.8 Points of Inflection

If $(c, f(c))$ is point of inflection on the graph of $f$, then either $f''(c) = 0$ or $f''(c)$ doesn't exist @ $x=c$

Ex 2: $f(x) = \cos x$ on $[0, 2\pi]$ find points of inflection

$f'(x) = -\sin x$ $\boxed{f''(x) = -\cos x}$ $0 = -\cos x$ → $\pi/2, 3\pi/2$

Point #1: $(\frac{\pi}{2}, 0)$ Point #2: $(\frac{3\pi}{2}, 0)$

The points of inflection are @ $(\frac{\pi}{2}, 0)$ & $(\frac{3\pi}{2}, 0)$.

Ex 3: $f(x) = 3x^5 - 5x^4 + 1$ → $f'(x) = 15x^4 - 20x^3$ $f''(x) = 60x^3 - 60x^2$

$0 = 60x^3 - 60x^2$ → $0 = 60x^2(x-1)$ → $0 = x^2$ → $0 = x$ $0 = x-1$ $1 = x$

$c = 0, 1$

$(-\infty, 0)(0,1)(1,\infty)$ point of inflection is $(1, -1)$.

$f''$: $-1$: $-$; $1/2$: $-$; $2$: $+$

$f(x)$ is concave down on $(-\infty, 1)$ & concave up on $(1, \infty)$.

Second Derivative Test (Theorem 3.9)

$f$ is a function such that $f'(c) = 0$ & $f''(x)$ exists on "I" including "c"
← $(x_1, x_2)$

- if $f''(c) > 0$ then $f$ has rel min @ $(c, f(c))$
- if $f''(c) < 0$ then $f$ has rel max @ $(c, f(c))$

If $f''(c) = 0$, the test fails; use the first derivative test.

Ex 4: $f(x) = x^3 - 12x - 5$ → $f'(x) = 3x^2 - 12$ → $0 = 3x^2 - 12$
$12 = 3x^2$ → $4 = x^2$
$x = \pm 2$

$f''(x) = 6x$ → $f''(2) = 6(2)$ → $+12 (>0)$ concave up → rel. min
$f''(-2) = 6(-2)$ → $-12$ concave down → rel. max.

There is a relative minimum of $-21$ @ $x=2$.
There is a relative maximum of " @ $x-2$.

HW:

# 3.5 Limits at Infinity (End Behavior) Oct. 31. 2024

Warm Up $f(x) = x^4 - x^3 - 3x^2 - 2$

$f'(x) = 4x^3 - 3x^2 - 6x \rightarrow f''(x) = 12x^2 - 6x - 6 \rightarrow 0 = 12x^2 - 6x - 6 \rightarrow x^2 - 6x - 72$

$0 = (x-12)(x+6) \rightarrow 0 = (x-1)(x+\frac{1}{2})$ $\cancel{x} = -\frac{1}{2}, 1$

$(-\infty, -\frac{1}{2})(-\frac{1}{2}, 1)(1, \infty)$ $f(-\frac{1}{2}) =$

| | $-1$ | $0$ | $3$ |
|---|---|---|---|
| $f''$ | $+$ | $-$ | $+$ |

Inflections: $(-\frac{1}{2}, -\frac{41}{16})$ & $(1, -5)$

Concave up on $(-\infty, -\frac{1}{2}) \cup (1, \infty)$. Concave down on $(-\frac{1}{2}, 1)$.

---

What is a limit @ infinity? End behavior of a function's graph (in limit notation)

Ex: left side $\lim\limits_{x \to -\infty} f(x) =$ right side $\lim\limits_{x \to \infty} f(x) =$

How do you find a horizontal asymptote?

for $f(x) = \frac{ax^n}{bx^m}$

- if $n < m$, horizontal asymptote is $y = 0$
- if $n = m$, horizontal asymptote is $y = \frac{a}{b}$
- if $n > m$, there is no horizontal asymptote
- if $n > m$ by exactly 1, there is a slant asymptote

Definition of a Horizontal Asymptote:

The line $y = L$ is a horizontal asymptote on "$f$" if

$\lim\limits_{x \to \infty} f(x) = L$ and/or $\lim\limits_{x \to -\infty} f(x) = L$

Ex 1:

1. H.A. is $y = 0$ $\lim\limits_{x \to \infty} f(x) = 0$
2. H.A. is $y = \frac{2}{5}$ $\lim\limits_{x \to -\infty} f(x) = \frac{2}{5}$
3. no H.A. $\lim\limits_{x \to \infty} f(x) = \infty$ $\frac{2x^5}{3x^2} \rightarrow \frac{2}{3}x^3$
4. no H.A $\lim\limits_{x \to -\infty} f(x) = -\infty$

for 3 & 4, divide leading terms to find term for polynomial rules of End Behavior

Theorem 3.10 Limits @ infinity

if "r" is positive & "c" is real, then $\lim_{x\to\infty} \frac{c}{x^r} = 0$.

And if $x^r$ is defined when $x<0$, then $\lim_{x\to-\infty} \frac{c}{x^r} = 0$.

Ex 2: $\lim_{x\to\infty}(5) - \lim_{x\to\infty}\left(\frac{2}{x^2}\right) \Rightarrow 5-0 = 5$

or translate: $\lim_{x\to\infty}\left(\frac{5x^2-2}{1x^2}\right) = 5$

Ex 3:

$\lim_{x\to\infty} \frac{2x-1}{1x+1} = 2$

even-even-odd $\begin{cases} \sqrt[4]{x^{12}} = x^{3} \\ \sqrt[4]{x^{12}} = \pm x \text{ or } |x| \end{cases}$

Ex 4:

ⓐ $\lim_{x\to\infty} \frac{3x-2}{\sqrt{2x^2+1}}$ HA: $y = \frac{3}{\sqrt{2}}$

if $\lim_{\to\infty}$, use +

ⓑ $\lim_{x\to-\infty} \frac{3x-2}{\sqrt{2x^2+1}}$ HA: $y = \frac{3}{-\sqrt{2}}$

if $\lim_{x\to-\infty}$, use −

$\sqrt{x^2} = \pm x$

Ex 5:

ⓐ $\lim_{x\to\infty} \sin x$ = DNE, due to oscillation

ⓑ $\lim_{x\to\infty} \frac{\sin x}{x} = 0$ squeeze theorem

202 HW: Q: 1-6, 13, 17, 21, 23, 27, 31, 33, 35

# 3.6 Curve Sketching

Sep. 5. 2024

WU: ① $\lim_{x \to -\infty} f(x) = +\infty$ ② $\lim_{x \to -\infty} f(x) = 2$

③ $\lim_{x \to \infty} f(x) = -\infty$ ④ $\lim_{x \to \infty} f(x) = \frac{3}{2}$ ⑤ $\lim_{x \to \infty} f(x) = \frac{\sqrt{3}}{2}$

Guidelines for Analyzing a function & How to Sketch it:

HW: 9, 14, 21 Start

- **x-int. & y-int.:**

x: set function equal to 0 & solve

y: set all x's to 0s in function, simplify

- **domain & range:**

D: Set of all x-vals in $f(x)$

R: Set of all y-vals in $f(x)$

- **Vertical asymptotes**

check for holes in function, if cannot be removed = VA

or check what makes denom. 0

- **relative extrema (local)**

minimum or maximum of $f(x)$ (find y-val)

- **Hole**

point that is undefined on $f(x)$

common factor between numerator & denominator

- **Point of inflection**

point where concavity switches from up to down or vice versa

- **symmetry:**

when a graph is mirrored over an axis:

→ flipped on y-axis

→ flipped on Origin

- **continuity:**

every x is defined, no removable or non-removable discontinuities

- **differentiability**

try to take derivative of function

no cusps

- **Horizontal Asymptote**

$n < m \quad y = 0$

$n = m \quad y = a/b$

$n > m \quad y =$ no HA

line $f(x)$ approaches a $x \to \pm\infty$

- **Concavity**

if $f''(x) > 0$ : concave up

if " $< 0$ : concave down

- **Limits @ infinity**

limit as $x$ approaches $\pm\infty$

$\lim_{x \to \pm\infty} f(x) =$ H.A. or $\lim_{x \to \pm\infty} f(x) =$ polynomial end behavior rules ex 1: $f(x) = x^3 - 4x$ $= x(x^2-4) \Rightarrow x = 0, \pm 2$

$f'(x) = 3x^2 - 4$ $f''(x) = 6x = 0$ x-int: $0, \pm 2$ y-int: 0

VA: no VA HA: no HA Critical numbers: $\pm\frac{2}{\sqrt{3}}$ (from $f'(x)$)

$(-\infty, -\frac{2}{\sqrt{3}})\ (-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}})\ (\frac{2}{\sqrt{3}}, \infty)$ relative minimum is $-3.079$ @ $x = \frac{2}{\sqrt{3}}$

$f'$: $-2$ / $+$ , $0$ / $-$ , $2$ / $+$ relative maximum is $3.079$ @ $x = -\frac{2}{\sqrt{3}}$

rel max rel min

$(-\infty, 0)\ (0, \infty)$ $f(x)$ is concave down on $(-\infty, 0)$.

$f''$: $-1$ / $-$ inflec at 0 $1$ / $+$ $f(x)$ is concave up on $(0, \infty)$.

con

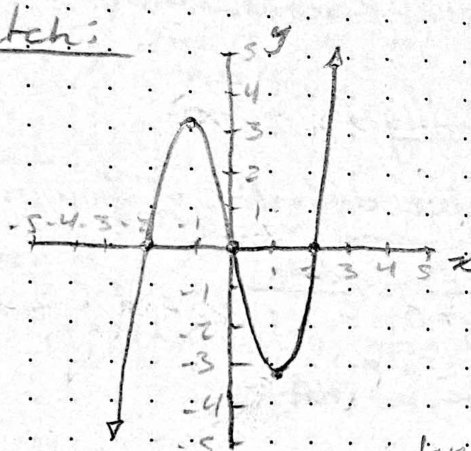
point of inflection is @ $(0,0)$

domain: $(-\infty, \infty)$. All real numbers Symmetry: $f(-x) = -x^3 - 4(-x)$

$= -x^3 + 4x$

$f(-x) = -f(x) \rightarrow$ odd!

Sketch:




if $b^2 - 4ac > 0$, there are real zeros

slant @

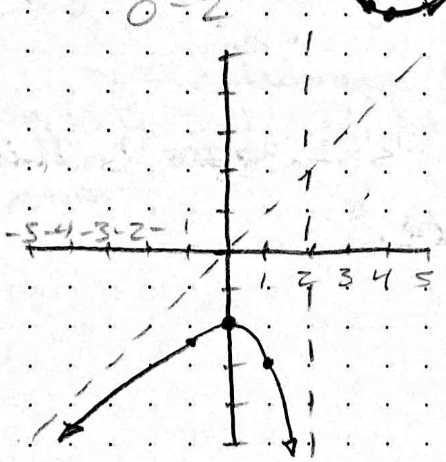
Ex 2: $f(x) = \frac{x^2 - 2x + 4}{x - 2}$ $f'(x)$: $\frac{(x-1)(2x-2) - (x^2-2x+1)(1)}{(x-2)^2}$

$x - 2 = 0 \rightarrow x = 2$

```
2| 1  -2  4
   ↓   2  0
   1   0  4
```

x-int! $(-2)^2 - 4(1)(9) < 0$ no real zeros

y-int: $\frac{0 - 0 + 4}{0 - 2}$ $= -2 \rightarrow (0, -2)$ VA: $x = 2$



| x | y |
|---|---|
| 1 | -3 |
| -1 | -2.33 |
| 3 | 7 |
| 4 | 6 |
| 3.5 | 6.166 |

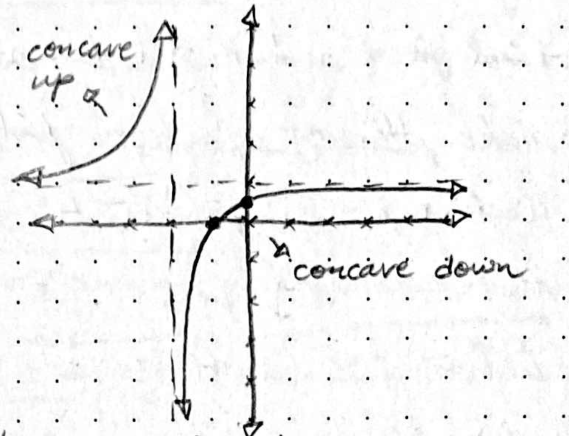
3.7 Optimization

Nov 12 2024

Warm-up $y = \frac{x+1}{x+2}$ $0 = x+1$

$x = -1$

$y = \frac{1}{2}$




$y' = \frac{x+2(1) - (x+1)(1)}{(x+2)^2}$

$y' = \frac{x+2-x-1}{(x+2)^2} \rightarrow y' = \frac{1}{(x+2)^2}$

$y'' = \frac{(x+2)^2(0) - (1)2(x+2)(1)}{(x+2)^4} \rightarrow \frac{-2x-4}{(x+2)^4} \rightarrow \frac{-2(x+2)}{(x+2)^2(x+2)^2} \rightarrow \frac{-2}{(x+2)^3}$

$0 = \frac{1}{(x+2)^2}$ no critical numbers

$x \neq -2$ $(-\infty, -2)(-2, \infty)$ no relative extrema

$f'$: $-3$: $+$, $0$: $+$

$0 = \frac{-2}{(x+2)^3} \rightarrow$ no critical values $(-\infty, -2)(-2, \infty)$

$x \neq -2$

$f''$: $-3$: $+$ up, $0$: $-$ down

*Check whether tests ask for extrema x-val. or actual point.

Optimization - The process of maximizing or minimizing

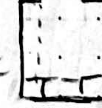
- for calculus: finding the mins. & maxs.

Ex1: Making a box, no top 5.5 in × 4.25 in sheet cardboard

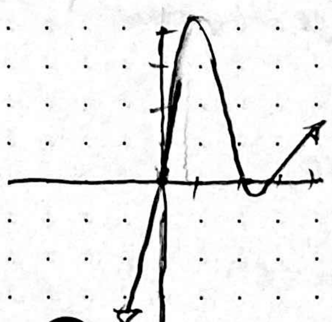
how big can the box be made if cutting squares of x size out of the corners?

$V = L \times W \times H$

$V = (5.5-2x) \times (4.25-2x) \times (x)$




$h = x$



max = 0.793 in

The squares should be 0.793 in long to hold a maximum volume of 8.269 in³.

Solving Minimum & Maximum Problems

1. Find given quantities to be determined. Make a sketch.
2. Make primary equation for quantity to be maximized or minimized
3. Reduce prim. equation to one with 1 independent variable
   → Maybe secondary equations
4. Determine feasible domain of prime equation. Determine values where problem makes sense
5. Determine max or min values through calc

Ex 2: 2 positive nums, $x$ & $y$

The product is 192 & the sum of $x$ & $3y$ is a min

$x \cdot y = 192$ $\qquad$ $\underbrace{x + 3y = \text{min}}_{\text{minimize}}$ on graph or "S"

$S(x) = x + 3y \rightarrow 192/x = y$

$S(x) = x + 3\left(\frac{192}{x}\right) \Rightarrow S(x) = x + 576x^{-1}$

Don't use $+\infty$ for this problem → feasible domain

$S'(x) = 1 - 1(576)x^{-2} \rightarrow S'(x) = 1 - \frac{576}{x^2}$

$0 = 1 - \frac{576}{x^2} \rightarrow 1 = \frac{576}{x^2} \rightarrow x^2 = 576 \rightarrow x = \pm 24$

use only $+24$ $\qquad x = 24$

| | $(0, 24)$ | $(24, 192)$ |
|---|---|---|
| | 1 | 25 |
| $S'(a)$ | $-$ | $+$ |

rel min.

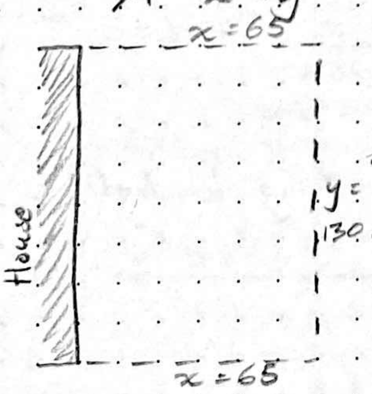
find $y \rightarrow \rightarrow y = 8$

$24 \cdot y = 192$
$y = 8$

The 2 positive numbers that minimize the sum are 24 & 8.

Ex 3: 260 ft material. Rect shape. Only 3 sides of rect

$A = x \cdot y$ $\qquad 260 = 2x + y$ $\qquad$ maximize the area




$A(x) = x(260 - 2x) \leftarrow 260 - 2x = y$

$A(x) = 260x - 2x^2$ $\qquad (0, 65)(65, 260)$

| | $(0, 65)$ | $(65, 260)$ |
|---|---|---|
| | 1 | 66 |
| $f'$ | $+$ | $-$ |

max @ $x = 65$

$A'(x) = 260 - 4x$

$0 = 260 - 4x$ $\qquad 260 = 130 + y$

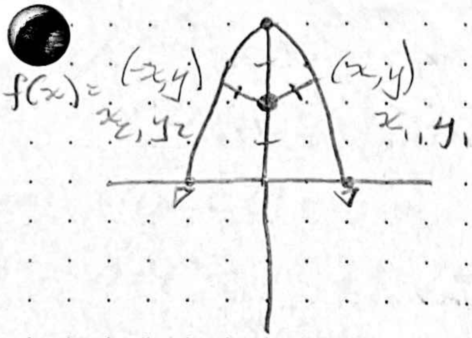
$4x = 260$ $\qquad y = 130$

$x = 65$

The dimensions of the fence that maximizes its area are 65 ft x 130 ft.

Ex 4: Which points on $y = 4 - x^2$ are closest to $(0,2)$?

minimize → closest $\quad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$




minimize distance

$$d = \sqrt{(x-0)^2 + (y-2)^2}$$

$$d = \sqrt{(x-0)^2 + ((4-x^2)-2)^2}$$

$$d = \sqrt{x^2 + (4-x^2-2)^2} \to \sqrt{x^2 + (2-x^2)^2}$$

$$d = \sqrt{x^4 - 3x^2 + 4}$$

$$\frac{d}{dx}\left(x^4 - 3x^2 + 4\right) = 4x^3 - 6x = 0$$

$$2x(2x^2 - 3) = 0$$

$x = 0 \qquad x = \pm\sqrt{\frac{3}{2}}$

$(-\infty, -\sqrt{\frac{3}{2}})\ (-\sqrt{\frac{3}{2}}, 0)\ (0, \sqrt{\frac{3}{2}})\ (\sqrt{\frac{3}{2}}, \infty)$

$- \qquad + \qquad - \qquad +$

min @ $x = \pm\sqrt{\frac{3}{2}}$

The closest points to $(0,2)$ are $\left(\pm\sqrt{\frac{3}{2}}, \frac{5}{2}\right)$.

HW: 4, 6, 9, 11, 16, 19, 25

Chapter 3 Review

① $f(x) = x^2+5x$, $[-4, 0]$ $f'(x) = 2x+5$ $0 = 2x+5 \Rightarrow -5 = 2x \Rightarrow x = -\frac{5}{2}$

$-4^2+5(-4) \Rightarrow 16-20 \Rightarrow -4$ ($x=-4$) $\quad$ $0$ ($x=0$) $\quad$ $(-\frac{5}{2})^2+5(-\frac{5}{2}) = -\frac{25}{4}$ ($x = -\frac{5}{2}$ or $-\frac{10}{4}$)

Absolute maximum of $0$ @ $x=0$.

Absolute minimum of $-\frac{25}{4}$ @ $x = -\frac{10}{4}$.

⑦ $g(x) = 2x + 5\cos x$, $[0, 2\pi]$ $g'(x) = 2 - 5\sin x$

$0 = 2-5\sin x \Rightarrow 5\sin x = 2 \Rightarrow \sin x = \frac{2}{5}$ $\quad$ $y = 0.88$, $x = 2.73$

$0+5(1) \Rightarrow 5$ ($x=0$) $\quad$ $4\pi+5\cos(2\pi) \Rightarrow 4\pi+5 \Rightarrow 4\pi+5 \Rightarrow 17.566$ ($x = 2\pi$)

Abs minimum of $0.88$ @ $x = 2.73$

Abs max of $17.566$ @ $x = 2\pi$

⑨ $f(x) = 2x^2-7$, $[0,4]$. $f'(x) = 4x$ $\quad$ $f(0) = 0-7 \Rightarrow -7$, $f(4) = 8-7 \Rightarrow 1$

$f(0) \neq f(4)$, Rolle's Theorem cannot be applied.

⑪ $f(x) = \frac{x^2}{1-x^2}$, $[-2, 2]$ $\quad$ $f(-2) = \frac{(-2)^2}{(1-(-2)^2)} \Rightarrow \frac{4}{-3} \Rightarrow -\frac{4}{3}$

$f(-2) = f(2)$ $\quad$ $f(2) = \frac{(2)^2}{1-(2)^2} \Rightarrow \frac{4}{-3} \Rightarrow -\frac{4}{3}$

$f'(x) = \frac{(1-x^2)(2x)-(x^2)(-2x)}{(1-x^2)^2} \Rightarrow \frac{2x-2x^3+2x^3}{(1-x^2)^2} \Rightarrow \frac{2x}{(1-x^2)^2}$

$0 = \frac{2x}{(1-x^2)^2} \Rightarrow 0 = 2x \Rightarrow x=0$ $\quad$ $f(0) = \frac{0}{1-0} \Rightarrow 0$ $\quad$ $c = (0,0)$

not continuous on $[-2,2]$, Rolle's theorem does not apply.

⑬ ~~$f(x) = x^{2/3}$, $[1, 8]$ ($a$, $b$) continuous ✓ $f'(x) = \frac{2}{3}x^{-1/3}$ diff'able ✓~~

~~$\frac{2}{3}x^{-1/3} = \frac{f(8)-f(1)}{8-1}$ $\quad$ $f(1) = 1$ $\quad$ $f(8) = 4$~~

~~$\frac{2}{3}x^{-1/3} = \frac{4-1}{8-1} \Rightarrow \frac{2}{3}x^{-1/3} = \frac{3}{7} \Rightarrow x^{-1/3} = \frac{3}{7}\cdot\frac{3}{2} \Rightarrow x^{-1/3} = \frac{9}{14}$~~

$\frac{1}{\sqrt[3]{x}} \Rightarrow 1 = \frac{9}{14}\sqrt[3]{x} \Rightarrow \frac{14}{9} = \sqrt[3]{x} \Rightarrow x = 3.764$

㉑ $f(x) = x^2+3x-12 \Rightarrow f'(x) = 2x+3$ $\quad$ $0 = 2x+3 \Rightarrow x = \frac{-3}{2}$

$(-\infty, -\frac{3}{2})(-\frac{3}{2}, \infty)$ $\quad$ $f(x)$ is increasing on $(-\frac{3}{2}, \infty)$ and decreasing on $(-\infty, -\frac{3}{2})$.

$f'$: $-2$: $-$ $\quad$ $0$: $+$

(27) $f(x)=x^2-6x+5 \to f'(x)=2x-6 \quad 0=2x-6 \to x=\frac{6}{2} \to x=3$

$(-\infty,3)(3,\infty)$ $f(x)$ is inc. on $(3,\infty)$ and dec. on $(-\infty,3)$.

$f'$ $\quad \overset{0}{-} \quad \overset{4}{+}$ there is a relative minimum of $-4$ @ $x=3$.

$(3)^2-6(3)+5 \to 9-18+5 \to 14-18 \to -4$

(29) $h(t)=\frac{1}{4}t^4-8t \to h'(t)=1t^3-8$

$0=t^3-8 \to t^3=8 \to t=2 \quad (-\infty,2)(2,\infty)$ $\quad h(2)=\frac{1}{4}(2)^4-8(2)$

$f' \quad \overset{0}{-} \quad \overset{3}{+}$ $\quad \to 4-16$

$h(x)$ is increasing on $(2,\infty)$ & decreasing on $(-\infty,2)$. There is a relative minimum of $-12$ @ $x=2$.

(31) $f(x)=\frac{x+4}{x^2} \to f'(x)=\frac{(x^2)(1)-(x+4)(2x)}{x^4} \to \frac{x^2-2x^2+8x}{x^4}$

$f'(x)=\frac{-x^2+8x}{x^4} \to 0=\frac{-x^2+8x}{x^4} \to -x^2+8x=0 \to x(-x+8)=0$

$\to x\neq 0$

~~$x=0$~~ $\quad -x+8=0 \to 8=x$
$\quad\quad +x \quad\quad +x$

(33) $f(x)=\cos x-\sin x,\ (0,2\pi)$

$f'(x)=-\sin x-\cos x \to 0=-\sin x-\cos x \to 0=\sin x+\cos x$

$x=\frac{3\pi}{4},\ \frac{7\pi}{4}$

sin = y-val

on unit circle!

# 4.1 Antiderivatives & Indefinite Integration

3 Dec 2024

WU ① $f(x) = x^3 + 2x^2 - 5 \rightarrow f'(x) = 3x^2 + 4x$

② $f(x) = (2x+5)^5 \rightarrow f'(x) = 5(2x+5)^4(2) \rightarrow f'(x) = 10(2x+5)^4$

Antiderivative – A function that reverses what a derivative does.

↳ "Integral" sometimes very hard to find

Definite Integral – Area under the function's curve

## Antiderivative

A function F is an antiderivative of f on an interval I when $F'(x) = f(x)$ for all x in I.

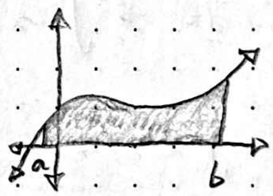
Indefinite Integral → no limits/boundaries, no a or b

$$\int f(x)\,dx = F(x) + C$$

↳ Ans is a function

Definite Integral → has a and b

$$\int_a^b f(x)\,dx = \text{Area / real number}$$




Ex:

$f(x) = x^3 + 4x + 1 \rightarrow f'(x) = 3x^2 + 4$

$f(x) = x^3 + 4x - 3$ All have the same derivative

$f(x) = x^3 + 4x + \frac{1}{2}$ $+ C$ → Know 1 more point to find original

Variable of integration Constant of integration

$$y = \int f(x)\,dx = F(x) + C$$

Integrand An antiderivative of f(x)

Ex 1: $y' = 2 \rightarrow y = 2x + C$

$dx \cdot \frac{dy}{dx} = 2 \cdot dx$

$\rightarrow \int (1)\,dy = \int 2\,dx$

$\rightarrow \boxed{y = 2x + C}$

**Power Rule**

$$\boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\quad n \neq -1}$$

$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x)$ Differentiation is the "inverse" of integration

Ex 2: $\int 3x^2\,dx \to 3\int x^2\,dx \to \frac{3(x^3)}{3} + C$

$\to x^3 + C = \int 3x^2\,dx$

Ex 3: find indefinite integral

(a) $\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = \frac{x^{-2}}{-2} + C$ or $\frac{1}{-2x^2} + C$

(b) $\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{2x^{3/2}}{3} + C$

(c) $\int 2\sin x\,dx = 2\int \sin x\,dx = -2\cos x + C$

Always $+C$ for indefinite integrals!

Ex 4:

(a) $\int dx \to \int 1\,dx \to \int 1x^0\,dx = x + C$

(b) $\int (x+2)\,dx = \int x\,dx + \int 2\,dx = \frac{x^2}{2} + 2x + C$

Ex 5:

$\int \frac{x+1}{\sqrt{x}}\,dx \to \frac{x+1}{\sqrt{x}} \to \frac{x+1}{x^{1/2}} \to (x+1)x^{-1/2} \to x^{1/2} + x^{-1/2}$

$= \int (x^{1/2} + x^{-1/2})\,dx = \frac{2x^{3/2}}{3} + 2x^{1/2} + C$

Ex 6: $\int \frac{\sin x}{\cos^2 x}\,dx \to \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} \to \tan x \cdot \sec x$

$\to \int \sec x \tan x\,dx = \sec x + C$

Ex 7:

(a) $\int \frac{2}{\sqrt{x}}\,dx \to 2x^{-1/2} \to \int 2x^{-1/2}\,dx = 4x^{1/2} + C$

(b) $\int (t^2+1)^2\,dt \to t^4 + 2t^2 + 1 \to \int (t^4 + 2t^2 + 1)\,dt = \frac{t^5}{5} + \frac{2}{3}t^3 + t + C$

(c) $\int \frac{x^3+3}{x^2}\,dx \to x + 3x^{-2} \to \int (x + 3x^{-2})\,dx = \frac{x^2}{2} + \frac{3x^{-1}}{-1} \to \frac{x^2}{2} - \frac{3}{x} + C$

ⓓ $\int \sqrt[3]{x}\,(x-4)\,dx = \int (x^{4/3} - 4x^{1/3})\,dx = \frac{3x^{7/3}}{7} - 4x^{4/3}\left(\frac{3}{4}\right) + C$

$$= \frac{3x^{7/3}}{7} - 3x^{4/3} + C$$

To find a particular solution,
you need an initial condition given as a point.

This means find $C$ using the initial condition.

Ex 8: $F'(x) = \frac{1}{x^2}$, $x > 0$ find general solution

$\frac{dy}{dx} = x^{-2} \rightarrow \int dy = \int x^{-2}\,dx \rightarrow y = \frac{x^{-1}}{-1} + C \rightarrow \boxed{y = -\frac{1}{x} + C} = F(x)$

find the particular solution that satisfies $F(1) = 0 \rightarrow (1, 0)$

$0 = \frac{-1}{1} + C \rightarrow 1 = C \rightarrow$ specific: $\boxed{y = -\frac{1}{x} + 1}$

Ex 9: Ball thrown up, initial velocity $\underset{V_0}{64}$ ft/s, initial height $\underset{S_0}{80}$ ft.

ⓐ find position function giving the height "s" as a function of time:

$s(t) = -16t^2 + 64t + 80$

$(s(t) = -16t^2 + V_0 t + S_0)$ = position func.

ⓑ When does the ball hit the ground?

$0 = -16t^2 + 64t + 80$

$0 = -16(t^2 - 4t - 5) \rightarrow 0 = t^2 - 4t - 5 \rightarrow (x-5)(x+1)$

negative time isn't an answer, eliminate

$\rightarrow x = 5, \cancel{-1} \rightarrow$ Ball hits the ground @ $t = 5$

HW: p 251; Qs: 5, 9, 13, 17, 21, 27, 33, 37, 50, 56

4.4.A Fundamental Theorem of Calculus  Dec. 5. 2024

WU ② $\int -\frac{7}{\cos^2 x}\,dx \rightarrow \int -7\sec^2 x\,dx \rightarrow -7\tan x + C$

⑨ $\int \frac{3\cos x}{\sin^2 x}\,dx \rightarrow 3\int \left(\frac{\cos x}{\sin x}\right)\left(\frac{1}{\sin x}\right)dx \rightarrow 3\int \cot x \csc x\,dx \rightarrow -3\csc x + C$

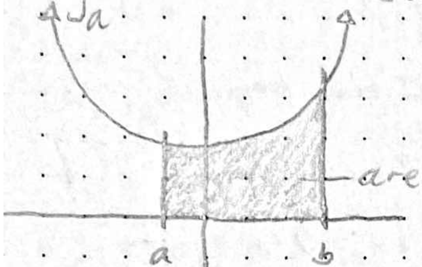
FTC (Fundamental...)

If function on $[a,b]$, & F is an antiderivative of f on $[a,b]$ the.

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad = \text{real \# (area)}$$

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a) = \text{real number}$$

no + c

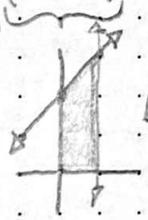



$\int_a^b f(x)\,dx$ = area of region enclosed by
- $f(x)$
- $x$-axis
- $x = a$
- $x = b$

Ex.1:

Trapazoid area: $\frac{1}{2} h(b_1 + b_2)$

$$\int_0^1 (x+2)\,dx \rightarrow \underbrace{\tfrac{1}{2}(1)(2+3)}^{\text{Trap. area (Geometry)}} \rightarrow \tfrac{1}{2}(5) \rightarrow \boxed{\tfrac{5}{2}}$$



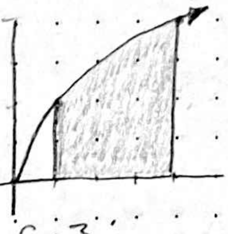
Calc way

$$\int_0^1 (x+2)\,dx = \left[\frac{x^2}{2} + 2x\right]_0^1 = \Big[\underbrace{(\tfrac{1}{2}+2)}_{f(b)} - \underbrace{(0)}_{f(a)}\Big] = \boxed{\tfrac{5}{2}}$$

Ex 2:

$$\int_1^4 2\sqrt{x}\,dx \rightarrow \left[2\left(\frac{x^{3/2}}{3/2}\right)\right] \rightarrow \left[2\left(\frac{2x^{3/2}}{3}\right)\right] \rightarrow \frac{4x^{3/2}}{3} \text{ or } \left[\frac{4}{3}x^{3/2}\right]_1^4$$

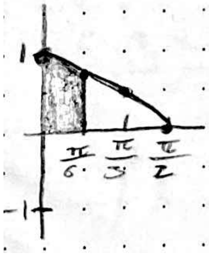
$2x^{1/2}$



$$= \frac{4}{3}(4)^{3/2} - \frac{4}{3}(1)^{3/2} \rightarrow \frac{32}{3} - \frac{4}{3} \rightarrow \boxed{\frac{28}{3}}$$

Ex 3:

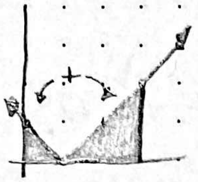
$$\int_0^{\pi/6} \cos x\,dx \rightarrow \Big[\sin x\Big]_0^{\pi/6} \rightarrow \Big[\underbrace{\tfrac{1}{2}}_{f(b)} - \underbrace{0}_{f(a)}\Big] \rightarrow \boxed{\tfrac{1}{2}}$$




$a \longleftrightarrow b$ "long time no see"
$\cancel{+C}$

**Ex4:** $\int_0^{\pi/4} \sec^2 x \, dx \to [\tan x]_0^{\pi/4} \to [(\overset{f(b)}{1} - \overset{f(a)}{(\tfrac{0}{1})})] \to \boxed{1}$

**Ex5:** $\int_0^3 |x-1| \, dx$ $\qquad y = a|bx-h|+k \quad \text{vertex} = (\frac{h}{b}, k)$



$\left.\begin{aligned} &\to \tfrac{1}{2}(1\cdot 1) \to \tfrac{1}{2} \\ &+\tfrac{1}{2}(2\cdot 2) \to 2 \end{aligned}\right\} \boxed{\frac{5}{2}}$

$$\int_0^1 (-x+1)\,dx + \int_1^3 (x-1)\,dx$$

$$\left[\frac{-x^2}{2} + x\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^3$$

$$\left[\left(-\frac{1}{2}+1\right) - 0\right] + \left[\left(\frac{9}{2} - 3\right) - \left(\frac{1}{2} - 1\right)\right] \to \frac{1}{2} + \frac{3}{2} = \boxed{\frac{5}{2}}$$

**Ex6:** $y = 2x^2 - 3x + 2$

$$\text{Area} = \int_0^2 (2x^2 - 3x + 2)\,dx \to \left[\frac{2x^3}{3} - \frac{3x^2}{2} + 2x\right]_0^2$$

$$\to \left[\left(\frac{2(8)}{3} - \frac{3(4)}{2} + 4\right) - 0\right] \to \left(\frac{16}{3} - \frac{12}{2} + 4\right) \to \frac{16}{3} - 6 + 4 \to \frac{16}{3} - \frac{6}{3} \to \boxed{\frac{10}{3}}$$

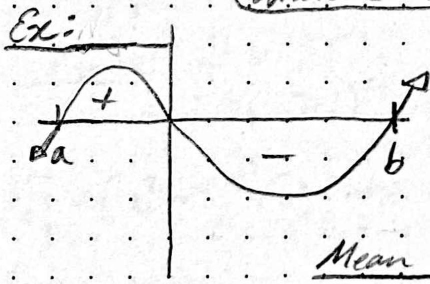
__Two Special Integrals:__

1. $\int_a^b f(x)\,dx = 0$ when $a = b$

2. $\int_a^b f(x)\,dx = \overset{\text{negate}}{-}c \qquad \int_b^a f(x)\,dx = c$

when $b > a$

**Ex:**



$\int_a^b f(x)\,dx < 0$ } when more area is under the $x$-axis than above it

## __Mean Value Theorem for Integrals__

If $f$ is continuous on $[a,b]$ and $c$ exists between $a$ & $b$ such that: $\int_a^b f(x)\,dx = f(c)(b-a)$

**Ex7:**

$f(x) = \sqrt{x}, \ [4,9] \qquad \int_4^9 \sqrt{x}\,dx = f(c)(b-a)$

$$\left[\frac{2x^{3/2}}{3}\right]_4^9 = \left(\frac{54}{3} - \frac{16}{3}\right) = \frac{38}{3} = f(c)(5) \to \frac{38}{3}\cdot\frac{1}{5} = f(c)$$

$$\frac{38}{15} = \sqrt{x}$$

$$\left(\frac{38}{15}\right)^2 = (\sqrt{x})^2$$

$x \approx 6.418$

$\boxed{c \approx 6.418}$

## Average Value of a function

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

If $f$ is continuous on $[a,b]$ then average value is

**Ex 8:** $f(x)=2x+1;\ [-2,3]$

$$\text{Average value} = \frac{1}{3+2}\int_{-2}^{3}(2x+1)\,dx = \frac{1}{5}\left[x^2+x\right]_{-2}^{3} \to \frac{1}{5}\left[(9+3)-(4-2)\right]$$

$$\to \frac{1}{5}[12-2] \to \frac{1}{5}(10) \to \boxed{2} = \text{Average value on } [-2,3]$$

## 4.4B Second Fundamental Theorem of Calculus:

$f$ is continuous on $I$, containing "a", for every $x$ in $I$:

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

$$\frac{d}{dx}\left[F(t)\right]_a^x = \frac{d}{dx}\left[F(x)-F(a)\right]$$
$$= f(x)+0 = f(x)$$

plug $x$ into "t"

$$\frac{d}{dx}\int_{\text{constant}}^{x} f(t)\,dx = f(x)$$

e.g.

① evaluate $\cancel{\frac{d}{dx}}\left[\cancel{\int_0^x}\sqrt{t^2+1}\,dx\right] = \sqrt{x^2+1}$

$$\frac{d}{dx}\int_{\text{constant}}^{g(x)} f(t)\,dt = f(g(x))\,g'(x)$$

e.g.

$$F(x)=\int_{\pi/2}^{x^3}\cos t\,dt = \cos(x^3)(3x^2)$$

$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt = f(g(x))\,g'(x) - f(h(x))\,h'(x)$$

4.5 Integration by Substitution

Dec, 10, 2024

WV: $\int_{\pi/4}^{\pi/2} -2\csc^2 x \, dx = \left[ 2\cot x \right]_{\pi/4}^{\pi/2} \to \left[ \left(2\cot\frac{\pi}{2}\right) - \left(2\cot\frac{\pi}{4}\right) \right] = \left(2\left(\frac{0}{1}\right)\right) - \left(2\,\frac{1}{1}\right) = -2$

## Antidifferentiation of Composite Functions

$\int f(g(x))g'(x) \, dx = F(g(x)) + C$ letting $u = g(x)$ gives $du = g'(x) \, dx$ and

$\int f(u) \, du = F(u) + C$

Let $g$ be a func, range on $I$; $f$ is a func., continuous on $I$. If $g$ is differentiable on it's domain & $F$ is an antiderivative of $f$ on $I$, then:

Recognizing the $f(g(x))g'(x)$ pattern;

Ex 1: $\int (x^2+1)^2(2x) \, dx$

$\hookrightarrow u = x^2+1 \to du = 2x \, dx \to \int \underbrace{(x^2+1)^2}_{u^2} \underbrace{(2x) \, dx}_{du} \to \int u^2 du \, dx \to$

$= \frac{u^3}{3} + C \to \frac{(x^2+1)^3}{3} + C$

Ex 2: $\int 5 \cos 5x \, dx$ $\quad u = 5x \quad du = 5 \, dx$

$\to \int \cos u \, du \to \sin u + C \to \boxed{\sin 5x + C}$

Ex 3: $\int x(x^2+1)^2 \, dx \to u = x^2+1 \quad du = 2x \, dx \to$

$\frac{1}{2}\int x(x^2+1)^2(2) \, dx \to \frac{1}{2}\int u^2 \, du = \frac{1}{2}\left(\frac{u^3}{2}\right) + C \to \boxed{\frac{(x^2+1)^3}{6} + C}$

Ex 4: $\int \sqrt{2x-1} \, dx \to u = 2x-1 \quad du = 2 \, dx \to$

$\frac{1}{2}\int u^{1/2} du \to \frac{1}{\cancel{2}}\left(\frac{\cancel{2}u^{3/2}}{3}\right) + C \to \frac{u^{3/2}}{3} + C = \boxed{\frac{(2x-1)^{3/2}}{3} + C}$

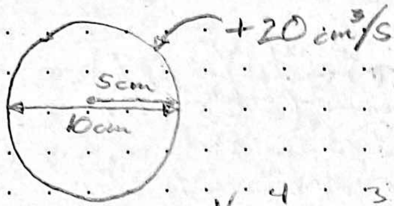
Ex 5: $\int (3-x^4)^6(4x^3) \, dx \to u = 3-x^4 \quad du = -4x^3 dx$

$-1\int u^6 du = -1\left(\frac{u^7}{7}\right) + C \to \frac{-u^7}{7} + C \to \boxed{\frac{-(3-x^4)^7}{7} + C}$

# Related Rates Practice

Balloon inflated at $+20\,cm^3/s$ (a) Find $\frac{dr}{dt}$ @ 5 cm r ✓

(Sphere) $\frac{dV}{dt}$ (b) Find $\frac{dA}{dt}$ @ 5 cm r.




$$V = \frac{4}{3}\pi r^3 \to \sqrt[3]{\frac{V}{\frac{4}{3}\pi}} = r$$

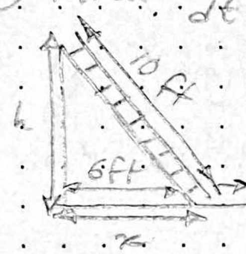
$$V = \frac{4}{3}\pi r^3 \to \frac{dV}{dt} = (4\pi r^2)\frac{dr}{dt} \to +20\,cm^3/s = 4\pi(5)^2\frac{dr}{dt}$$

$$\to 20 = 4\pi 25\frac{dr}{dt} \to 20 = 100\pi\frac{dr}{dt} \to \frac{dr}{dt} = \frac{20}{100\pi} \to \frac{dr}{dt} = \boxed{\frac{1}{5\pi}\,cm/s}$$ (a)

$$A = 4\pi r^2 \to \frac{dA}{dt} = 8\pi r\frac{dr}{dt} \to \frac{dA}{dt} = 8\pi(5)\left(\frac{1}{5\pi}\right) \to \boxed{\frac{dA}{dt} = 8\,cm^2/s}$$ (b)

Ladder (10 ft) against wall, bottom sliding away @ 2 ft/s.

(a) Find $\frac{dh}{dt}$ @ $x = 6$ ft.




$2\,ft/s$ $\left(\frac{dx}{dt} = 2\,ft/s\right)$

$h =$

$$x^2 + h^2 = 10^2$$

$$\frac{d}{dt}(x^2 + h^2) = \frac{d}{dt}10^2$$

$$\frac{2x\frac{dx}{dt} + 2h\frac{dh}{dt}}{2} = \frac{0}{2}$$

$$\to x\frac{dx}{dt} + h\frac{dh}{dt} = 0 \to h\frac{dh}{dt} = -x\frac{dx}{dt} \to \boxed{\frac{dh}{dt} = -\frac{x}{h}\cdot\frac{dx}{dt}}$$

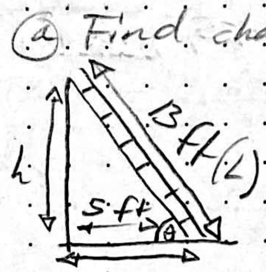
sub $x = 6$ for $h \to 6^2 + h^2 = 10^2 \to 36 + h^2 = 100 \to h^2 = 64 \to h = \boxed{8}$

$$\to \frac{dh}{dt} = -\frac{6}{8}\cdot 2\,ft/s \to \frac{dh}{dt} = -\frac{12}{8} = \boxed{-\frac{3}{2}\,ft/s}$$ (a)

Ladder (13 ft) against wall, bottom sliding away @ 3 ft/s $\left(\frac{dx}{dt}\right)$

(a) Find change in ∠ between ladder & ground when $x = 5$ ft

$\frac{d\theta}{dt}$




$$\cos\theta = \frac{x}{L} \to \cos\theta = \frac{x}{13} \to -\sin\theta\frac{d\theta}{dt} = \frac{1}{13}\frac{dx}{dt}$$

$$\to \frac{d\theta}{dt} = -\frac{1}{13\sin\theta}\frac{dx}{dt}$$

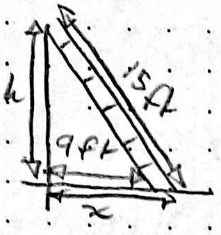
$$5^2 + h^2 = 13^2 \to 25 + h^2 = 169$$

$$\to h^2 = 144 \to h = 12\,ft$$

$$\to \sin\theta = \frac{12}{13} \to \frac{d\theta}{dt} = -\frac{1}{\frac{13}{1}\cdot\frac{12}{13}}(3\,ft/s) \to -\frac{1}{12}(3\,ft/s)$$

$$\to -\frac{3}{12} \to -\frac{1}{4} = \frac{d\theta}{dt} \to -\frac{1}{4}\,\text{radians}/s = \frac{d\theta}{dt}$$

15 ft, away @ 2ft/s ($\frac{dx}{dt}$) find $\frac{dh}{dt}$ when $x = 9$ ft



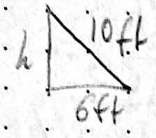

$$h^2+9^2=15^2 \to h^2+81=225 \to h^2=144 \to \boxed{h=12}$$

$$h^2+x^2=15^2 \to 2h\frac{dh}{dt}+2x\frac{dx}{dt}=0 \to 2h\frac{dh}{dt}=-2x\frac{dx}{dt}$$

$$\to h\frac{dh}{dt}=-x\frac{dx}{dt} \to \frac{dh}{dt}=-\frac{x}{h}\frac{dx}{dt} \to -\frac{9}{12}(2\ ft/s)$$

$$\to \frac{dh}{dt}=-\frac{18}{12} \to -\frac{9}{6}\ \boxed{-\frac{3}{2}\ ft/s}$$

10 ft, away @ 1 ft/s. find $\frac{dh}{dt}$ @ 6 ft




$$h^2+6^2=10^2 \to h^2+36=100 \to h^2=64 \to h=\boxed{8}$$

$$h^2+x^2=10^2 \to 2h\frac{dh}{dt}+2x\frac{dx}{dt}=0 \to \cancel{2}h\frac{dh}{dt}=-\cancel{2}x\frac{dx}{dt}$$

$$\frac{dh}{dt}=-\frac{x}{h}\frac{dx}{dt} \to \frac{dh}{dt}=-\frac{6}{8}\cdot(1\ ft/s) \to \frac{dh}{dt}=\boxed{-\frac{3}{4}\ ft/s}$$

Fundamental theorem of Calc, II

$$\frac{d}{dx}\left[\int_c^x f(t)\,dx\right]=f(x), \qquad \frac{d}{dx}\left[\int_c^{g(x)} f(t)\,dx\right]=f(g(x))g'(x)$$

$$\frac{d}{dx}\left[\int_{h(x)}^{g(x)} f(t)\,dx\right]=(f(g(x))g'(x))-(f(h(x))h'(x))$$

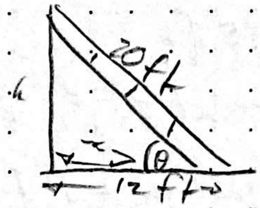
Derivatives of Trig funcs

$\frac{d}{dx}\cos x=-\sin x$ $\qquad$ $\frac{d}{dx}\sin x=\cos x$

$\frac{d}{dx}\sec x=\sec x\tan x$ $\qquad$ $\frac{d}{dx}\csc x=-\csc x\cot x$

$\frac{d}{dx}\tan x=\sec^2 x$ $\qquad$ $\frac{d}{dx}=\cot x=-\csc^2 x$

20 ft, find $\frac{d\theta}{dt}$ @ $x = 12$ ft $\qquad$ no $\frac{dx}{dt}$ given, ingnore




$$12^2+h^2=20^2 \to 144+h^2=400 \to h^2=256\ \boxed{h=16}$$

$$\cos\theta=\frac{x}{20} \to -\sin\theta\frac{d\theta}{dt}=\frac{1}{20}\frac{dx}{dt}$$

$$\to \frac{d\theta}{dt}=\frac{1}{20(-\sin\theta)}\frac{dx}{dt} \to \frac{d\theta}{dt}=\frac{-1}{20(\frac{16}{20})}\frac{dx}{dt}$$

$$\to -\frac{1}{16} \to \boxed{\frac{d\theta}{dt}=-\frac{1}{16}\ \text{radians}/ft}$$

Explicit Differentiation Practice

$2y^2 - 6 = y\sin x$ — $\boxed{\text{for } y>0}$

(a) show that $\frac{dy}{dx} = \frac{y\cos x}{4y - \sin x}$ $\rightarrow \frac{d}{dx}(2y^2 - 6) = \frac{d}{dx}(y\sin x)$

$\rightarrow 4y\frac{dy}{dx} = \left(\frac{dy}{dx}\cdot\sin x\right) + (y\cdot\cos x) \rightarrow \frac{dy}{dx}(4y - \sin x) = y\cos x$

$\rightarrow \boxed{\frac{dy}{dx} = \frac{y\cos x}{4y - \sin x}}$

(b) Make tangent to curve @ $(0, \sqrt{3})$

$\left.\frac{dy}{dx}\right|_{(0,\sqrt{3})} = \frac{\sqrt{3}\cos(0)}{4(\sqrt{3}) - \sin(0)} \rightarrow \frac{\sqrt{3}(1)}{4\sqrt{3} - 0} \rightarrow \frac{\sqrt{3}}{4\sqrt{3}} \rightarrow \boxed{\frac{1}{4}} = m \Big|_{(0,\sqrt{3})}$

$\boxed{y - \sqrt{3} = \frac{1}{4}(x)}$

(c) find $\frac{dy}{dx} = 0$ for $0 \le x \le \pi$

$\frac{y\cos x}{4y - \sin x} = 0 \rightarrow y\cos x = 0 \rightarrow \cos x = 0 \rightarrow \boxed{x = \frac{\pi}{2}} \rightarrow \boxed{\left(\frac{\pi}{2}, 2\right)}$

$y = 0$

$2y^2 - 6 = y\left(\sin\frac{\pi}{2}\right) \rightarrow 2y^2 - 6 = y \rightarrow 2y^2 - y - 6 = 0$

$\rightarrow y^2 - y - 12 \rightarrow \left(y - \frac{4}{2}\right)\left(y + \frac{3}{2}\right) \rightarrow (2y+3)(y-2) = 0$

$\rightarrow 2y + 3 = 0 \rightarrow 2y = -3 \rightarrow \cancel{y = -3/2}$ $y > 0$ $\quad y - 2 = 0 \rightarrow \boxed{y = 2}$

(d) does f have rel min, max, or neither @ $\left(\frac{\pi}{2}, 2\right)$

$$\frac{d^2y}{dx^2} = \frac{dy}{dx}\left(\frac{y\cos x}{4y - \sin x}\right) \rightarrow \frac{\left((4y - \sin x)\left(y(-\sin x) + \left(\left(\frac{dy}{dx}\right)(\cos x)\right)\right)\right) - \left((y\cos x)\left(4\frac{dy}{dx} - \cos x\right)\right)}{(4y - \sin x)^2}$$

$$\left.\frac{d^2y}{dx^2}\right|_{\left(\frac{\pi}{2}, 2\right)} \frac{\left(4(2) - \sin\frac{\pi}{2}\right)\left(2\left(-\sin\frac{\pi}{2}\right) + \left(\left(\frac{2\cos\frac{\pi}{2}}{4(2) - \sin\frac{\pi}{2}}\right)\left(\cos\frac{\pi}{2}\right)\right)\right) - \left(2\left(\cos\frac{\pi}{2}\right)\right)\left(4\left(\frac{2\cos\frac{\pi}{2}}{4(2) - \sin\frac{\pi}{2}}\right) - \left(\cos\frac{\pi}{2}\right)\right)}{\left(4(2) - \sin\frac{\pi}{2}\right)^2}$$

$$\rightarrow \frac{(8-1)(2(-1)) + \cancel{\left(\frac{2(0)}{8-1}\right)(0)} - \cancel{\left(2(0)\right)\left(4\left(\frac{2(0)}{8-1}\right)\right)} - 0}{(8-1)^2}$$

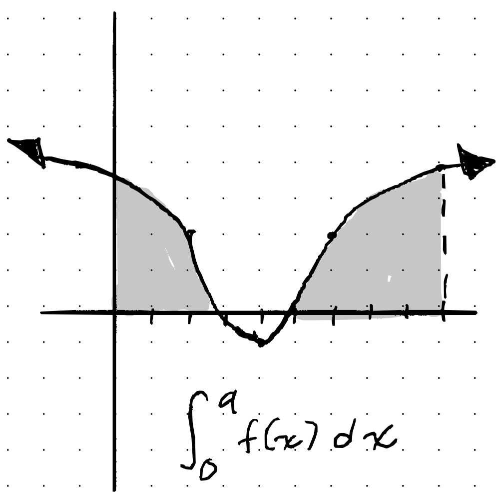
$\rightarrow \frac{-14}{49} \rightarrow -C \rightarrow$ $\boxed{\text{rel. max.}}$

@ $\left(\frac{\pi}{2}, 2\right)$

# AP Calculus AB 2nd Semester

$$\frac{d}{dx}\left(f(x)\right)$$

$$\int_a^b f(x)\,dx$$



$$\int_0^a f(x)\,dx$$

# 4.5 Integration by Substitution

## Antidifferentiation of Composite Functions:

Let "g" be a function, with a range of "I"; "f" is also a function which is continuous on "I". If "g" is differentiable on it's domain & "F" is an antiderivative of "f" on "I", then:

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C$$

$$\int f(u)\,du = F(u) + C$$

---

**Ex 1:** $\int (x^2+1)^2(2x)\,dx$ Recognizing the $f(g(x))g'(x)$ pattern:

$\rightarrow u = x^2+1 \rightarrow du = 2x\,dx \rightarrow \int \underbrace{(x^2+1)}_{u}{}^{2}\underbrace{(2x)\,dx}_{du} \rightarrow \int u^2\,du \rightarrow$

$= \frac{u^3}{3} + C \rightarrow \frac{(x^2+1)^3}{3} + C$

---

**Ex 2:** $\int 5\cos 5x\,dx$ $u = 5x$ $du = 5\,dx$

$\rightarrow \int \cos u\,du \rightarrow \sin u + C \rightarrow \boxed{-\sin 5x + C}$

---

**Ex 3:** $\int x(x^2+1)^2\,dx \rightarrow$ $u = x^2+1$ $du = 2x\,dx$

$\frac{1}{2}\int x(x^2+1)^2(2)\,dx \rightarrow \frac{1}{2}\int u^2\,du = \frac{1}{2}\left(\frac{u^3}{3}\right) + C \rightarrow \boxed{\frac{(x^2+1)^3}{6} + 3}$

---

**Ex 4:** $\int \sqrt{2x-1}\,dx \rightarrow$ $u = 2x-1$ $du = 2\,dx \rightarrow$

$\frac{1}{2}\int u^{1/2}\,du \rightarrow \frac{1}{2}\left(\frac{2u^{3/2}}{3}\right) + C \rightarrow \frac{u^{3/2}}{3} + C = \boxed{\frac{(2x+1)^{3/2}}{3} + C}$

---

**Ex 5:** $\int (3-x^4)^6(4x^3)\,dx \rightarrow u = 3-x^4$ $du = -4x^3\,dx$

$-1\int u^6\,du = -1\left(\frac{u^7}{7}\right) + C \rightarrow \frac{-u^7}{7} + C \rightarrow \boxed{\frac{-(3-x^4)^7}{7} + C}$

# Warm up and review Jan 14 2025

(1) $\int_1^9 \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx \to u = 1+\sqrt{x}$, $x^{1/2} \to \frac{1}{2\sqrt{x}}$, $du = \frac{1}{2\sqrt{x}}dx$

$2\int \frac{1}{u^2}du$

u-limits:
$x=1$
$u = 1+\sqrt{1} = 2$
$x=9$
$u = 1+\sqrt{9} \to 1+3 = 4$

$$2\int_2^4 u^{-2}du \to 2\left[\frac{u^{-1}}{-1}\right]_2^4 \to -2\left[\frac{1}{u}\right]_2^4$$

$$-2\left[\frac{1}{4}-\frac{1}{2}\right] \to -2\left[-\frac{1}{4}\right] \to \frac{1}{2}$$

(2) $\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx \to u = 2x-1 \to 2$, $\frac{du}{dx} = 2 \to du = 2dx$

$\to x = \frac{1+u}{2} \to \frac{1}{2}\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx(2)$

u-limits:
$x=1$
$2(1)-1 = 1$
$x=5$
$2(5)-1 = 9$

$$\to \frac{1}{2}\int_1^9 \frac{\left(\frac{u+1}{2}\right)}{u^{1/2}}\,du \to \frac{1}{4}\int_1^9 \left(\frac{u+1}{u^{1/2}}\right)du$$

$$= \frac{1}{4}\int_1^9 \left(u^{1/2}+u^{-1/2}\right)du \to \frac{1}{4}\left[\frac{2u^{3/2}}{3}+2\sqrt{u}\right]_1^9$$

$$\to \frac{1}{4}\left[\left(\frac{54}{3}+6\right)-\left(\frac{2}{3}+2\right)\right] \to \frac{1}{4}\left(\frac{72}{3}-\frac{8}{3}\right) = \frac{1}{4}\left(\frac{64}{3}\right)$$

$\left(\frac{2}{3}+2\right) \to \frac{8}{3}$

$\to \frac{16}{3}$

## Review of Final from December

$$\sin(-u) = -\sin(u)$$

learn to find limits in functions using conjugates

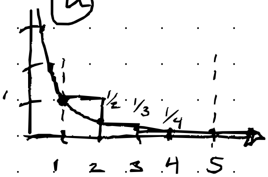
# 4.2 Areas

Jan 16

Warm up: $f(x) = \frac{1}{x}$, $[1, 5]$

mid point: 1.575
right: 1.283

ⓐ



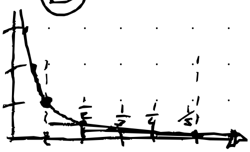

$(1\times 1) + (1\times \frac{1}{2}) + (1\times \frac{1}{3}) + (1\times \frac{1}{4})$

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

$\rightarrow 1 + \frac{3}{4} + \frac{1}{3} \rightarrow 1 + \frac{13}{12}$

$\frac{9}{12} + \frac{4}{12}$ $2 + \frac{1}{12} \rightarrow \frac{25}{12}$

ⓑ




$A = \int_1^5 \left(\frac{1}{x}\right) dx$

$\rightarrow = 1(f(2) + f(3) + f(4) + f(5))$

$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$

$= \frac{77}{60}$ 1.283

ⓐ 2.083

ⓑ 1.283

lower sum (inscribed) < actual area < upper sum (circumscribed)

## What is a sigma notation?

$$\text{sum of} \left\{ \sum_{i=\ldots}^{n} \text{expression in terms of } i \right.$$

$n \rightarrow$ last # to plug in?

$i = \ldots \rightarrow$ initial # to plug in

★ The sum of $n$ terms $a_1, a_2, a_3, \ldots\ldots, a_n$ is written as:

$$\sum_{i=1}^{n} a_i = a_1, a_2, a_3, \ldots a_n$$

$i$ = index of summation

## Ex 1:

ⓐ $\sum_{i=1}^{6} i = 1 + 2 + 3 + 4 + 5 + 6 = 21$

ⓑ $\sum_{i=0}^{5} (i+1) = (0+1) + (1+1) + (2+1) + (3+1) + (4+1) + (5+1) = 21$

(c) $\sum_{j=3}^{7} j^2 = 3^2 + 4^2 + 5^2 + 6^2 + 7^2 = 135$

(d) $\sum_{j=1}^{5} \frac{1}{\sqrt{j}} = \frac{1}{1} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} = 3.232$

## Properties of Summation

① $\sum_{i=1}^{n} k a_i = k \sum_{i=1}^{n} a_i$

② $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

## Summation Formulas

① $\sum_{i=1}^{n} c = cn$, c is a constant

② $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

③ $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

④ $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

**Ex 2:** $\sum_{i=1}^{n} \frac{i+1}{n^2}$ for $n = 10, 100, 1000, 10.000$

$= \frac{1}{n^2} \sum_{i=1}^{n} (i+1) = \frac{1}{n^2} \left( \sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \right)$

$\rightarrow \frac{1}{n^2} \left( \frac{n(n+1)}{2} + n \right) \rightarrow$ simplify $\rightarrow \frac{1}{n} \left( \frac{(n+1)}{2} + 1 \right) \cdot \frac{2}{2}$

$\rightarrow \frac{1}{n} \left( \frac{n+3}{2} \right) \rightarrow \boxed{\frac{n+3}{2n}}$

sum

$n = 10 \rightarrow \frac{10+3}{2(10)} \rightarrow \frac{13}{20}$

$n = 100 \rightarrow \frac{100+3}{2(100)} \rightarrow \frac{103}{200}$

# Upper Sums & Lower Sums

The sum of areas of the inscribed rectangles is called a lower sum, & the sum of the areas of the circumscribed rectangles is called an upper sum

$$\text{Lower sum} = s(n) = \sum_{i=1}^{n} f(m_i)\Delta x$$

$$\text{Upper sum} = S(n) = \sum_{i=1}^{n} f(M_i)\Delta x$$

$m_i = a + (i-1)\Delta x \qquad M_i = a + i\Delta x$

$$\Delta x = \frac{b-a}{n}$$

---

**Ex 3:** Find upper & lower sums for region bounded by $f(x) = x^2$ & the x-axis between $[0, 2]$

$$\underline{\text{Upper sum}} = \sum_{i=1}^{n} f(M_i)\Delta x$$

$\Delta x = \frac{2-0}{n} = \frac{2}{n}$

$M_i = 0 + i\left(\frac{2}{n}\right) = \frac{2i}{n}$

$$\rightarrow \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\frac{2}{n} = \frac{2}{n}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)^2$$

$$\rightarrow \frac{2}{n}\sum_{i=1}^{n}\left(\frac{4i^2}{n^2}\right) \rightarrow \frac{8}{n^3}\sum_{i=1}^{n}(i^2)$$

$$= \frac{\overset{4}{8}}{\underset{3n^2}{n^3}}\left(\frac{\cancel{n}(n+1)(2n+1)}{\underset{3}{6}}\right) \rightarrow \frac{4(n+1)(2n+1)}{3n^2}$$

$$\rightarrow \frac{4(2n^2+3n+1)}{3n^2} \rightarrow \boxed{\frac{8n^2+12n+4}{3n^2}} \quad \text{Upper Sum}$$

# Lower Sum $\underset{a}{[0},\underset{b}{2]}$ $f(x)=x^2$

lower sum $= \sum_{i=1}^{n} f(m_i)\,\Delta x$

$\Delta x = \frac{b-a}{n} = \frac{2-0}{n}$

$\rightarrow \frac{2}{n}$

$m_i = a+(i-1)\,\Delta x$

$= 0 + (i-1)\,\frac{2}{n}$

$= \frac{2(i-1)}{n}$

$= \sum_{i=1}^{n} f\left(\frac{2(i-1)}{n}\right)\frac{2}{n}$

$= \frac{2}{n}\sum_{i=1}^{n}\left(\frac{2(i-1)}{n}\right)^2$

$\left(\frac{2(i-1)}{n}\right)^2$

$\frac{2}{n}\cdot\frac{4}{n^2}$

$\frac{4}{n^2}(i^2-2i^2+1)$

$= \frac{8}{n^3}\sum_{i=1}^{n}(i^2-2i^2+1) = \frac{8}{n^3}\left(\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right)$

$\rightarrow \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6} - 2\left(\frac{n(n+1)}{2}\right) + n\right)$

$\rightarrow \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6} - 2\left(\frac{n(n+1)}{2}\right) + \frac{n}{1}\right)$

$\rightarrow \frac{8}{n^2}\left(\frac{(n+1)(2n+1)}{6} - 2\left(\frac{(n+1)}{2}\right) + \frac{1}{1}\right)$

$\rightarrow \frac{8}{n^2}\left(\frac{(n+1)(2n+1)}{6} - \frac{6(n+1)}{6} + \frac{6}{6}\right)$ $3n$

$\rightarrow \frac{8}{n^2}\left(\frac{2n^2+\overbrace{n+2n}+1}{6} - \frac{6n+6}{6} + \frac{6}{6}\right)$

$\rightarrow \frac{8}{n^2}\left(\frac{2n^2-3n+1}{6}\right) = \frac{16n^2-24n+8}{6n^2} \rightarrow \boxed{\frac{8n^2-12n+4}{3n^2}}$

---

$\lim_{n\to\infty}(S_n) = \frac{8}{3}$

Warm up Jan 17

42 $\int_0^3 f(x)\,dx = 4$ & $\int_3^6 f(x)\,dx = -1$

ⓐ $\int_0^6 f(x)\,dx = 3$ ✓

ⓑ $\int_6^3 f(x)\,dx = 1$ ✓

ⓒ $\int_3^3 f(x)\,dx = 0$ ✓

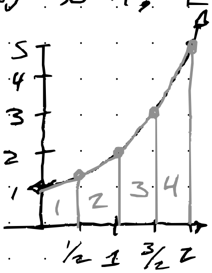
ⓓ $\int_3^6 -5f(x)\,dx = -5(-1) = 5$ ✓

# 4.6 Trapezoidal Rule

Learn the Trapezoidal Rule:

Find the area under the curve using 4 trapezoids

$f(x)=x^2+1;\ [0,2]$




$$A=\int_0^2 f(x^2+1)\,dx$$

$$\rightarrow \underbrace{\frac{1}{2}\left(\frac{1}{2}\right)\left(f(0)+f\left(\frac{1}{2}\right)\right)}_{\text{Trap. 1}} \rightarrow \frac{1}{4}\left(1+\left(\left(\frac{1}{2}\right)^2+1\right)\right)$$

$$+\frac{1}{2}\left(\frac{1}{2}\right)\left(f\left(\frac{1}{2}\right)+f(1)\right)$$

$$+\frac{1}{2}\left(\frac{1}{2}\right)\left(f(1)+f\left(\frac{3}{2}\right)\right)$$

$$+\frac{1}{2}\left(\frac{1}{2}\right)\left(f\left(\frac{3}{2}\right)+f(2)\right)\rightarrow$$

$$\frac{1}{4}\left[\left(f(0)+f\left(\frac{1}{2}\right)\right)+\left(f\left(\frac{1}{2}\right)+f(1)\right)+\left(f(1)+f\left(\frac{3}{2}\right)\right)+\left(f\left(\frac{3}{2}\right)+f(2)\right)\right]$$

$$A = \frac{1}{2}h(b_1+b_2)$$

$$\Delta x\,(f_{left}+f_{right})$$

$f(0)=1$ $\quad f\left(\frac{1}{2}\right)=5/4$

$f(1)=2$ $\quad f(3/2)=13/4$

$f(2)=5$

$$\rightarrow \frac{1}{4}\left(f(0)+2f\left(\frac{1}{2}\right)+2f(1)+2f(3/2)+f(2)\right)$$

$$\rightarrow \frac{1}{4}\left(1+2\left(5/4\right)+2(2)+2\left(13/4\right)+5\right)$$

$$\rightarrow \frac{1}{4}(19) \rightarrow \frac{19}{4}$$

## The Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)+\cdots+f(x_n)\right]$$

As "n" approaches $\infty$, leftside approaches $\int_a^b f(x)\,dx$

<u>Always Remember:</u>

1. You can only use the trapezoidal rule if $f(x)$ is given
2. All trapezoides must be the same width.

If these rules aren't met, use simple geometry:

$$A_{trap.} = \frac{1}{2}\Delta x(b_1 + b_2)$$

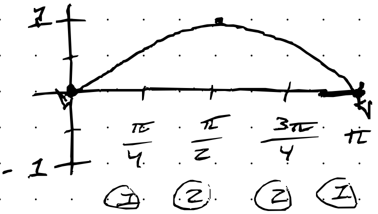
<u>Ex 1:</u>

use $n=4$

$$\int_0^{\pi} \sin x \, dx \approx \frac{\pi - 0}{8}\left[f(0) + 2f\left(\frac{\pi}{2}\right) + 2f\left(\frac{3\pi}{4}\right) + f(\pi)\right]$$

$$\to \frac{\pi}{8}\left[0 + \sqrt{2} + 2 + \sqrt{2} + 0\right]$$

$$\to \frac{\pi}{8}\left[2 + 2\sqrt{2}\right] \approx 1.896$$



# 4.3 Riemann Sums

Jan 21

**Warm up**

ⓐ $\sum_{i=2}^{5} i^2 - 5 \to \left((2^2-5)+(3^2-5)+(4^2-5)+(5^2-5)\right)$

$\to -1+4+11+20 \to 14+20 \to 34$

ⓑ $\sum_{i=2}^{4} 3^i \to \left((3^2)+3^3+3^4\right) \to 9+27+81$

$\to 81+36 \to 117$

ⓒ $\sum_{i=1}^{n} \frac{1}{n^3}\left(i^2+1\right) \to \frac{i^2+1}{n^3} \to \frac{1}{n^3}\sum_{i=1}^{n} i^2+1$

$\to \frac{1}{n^3}\left(\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} 1\right) \to \frac{1}{n^3}\left(\left(\frac{n(n+1)(2n+1)}{6}\right)+\frac{n}{1}\right)$

$\to \frac{1}{n^2}\left(\frac{(n+1)(2n+1)}{6}+\frac{1}{1}\right) \to \frac{(n+1)(2n+1)+6}{6n^2} \to \frac{2n^2+3n+7}{6n^2}$

**Ex 2:** $\int_{-2}^{1} 2x\,dx$ (use upper sum): $\lim_{n\to\infty}(S(n))$

$\to f(x)=2x \quad \Delta x=\frac{3}{n} \quad M_i=-2+i\Delta x \to -2+\frac{3i}{n}$

$\to \sum_{i=1}^{n} f(M_i)\Delta x = \sum_{i=1}^{n} f\left(-2+\frac{3i}{n}\right)\frac{3}{n}$

$\to \frac{3}{n}\sum_{i=1}^{n}\left(2\left(-2+\frac{3i}{n}\right)\right) \to \frac{3}{n}\left(\sum_{i=1}^{n}(-4)+\frac{6}{n}\sum_{i=1}^{n} i\right)$

$\to \frac{3}{n}\left(-4n+\frac{6}{n}\left(\frac{n(n+1)}{2}\right)\right) \to \frac{3}{n}\left(-4n+3n+3\right)$

$\to \frac{-12n+9n+9}{n} \to \lim_{n\to\infty}\frac{-3n+9}{n} \to \boxed{-3}$

$\int_{-2}^{1} 2x\,dx \to \left[x^2\right]_{-2}^{1} \to \left(1-(-2)^2\right) \to 1-4 \to -3$

# AP Question

| $t$ (hours) | 0 | 1 | 3 | 6 | 8 |
|---|---|---|---|---|---|
| $R(t)$ (liters/hr) | 1340 | 1190 | 950 | 740 | 700 |

$W(t) = 2000e^{-t^2/20}$ on $0 \le t \le 8$ (t in hours)

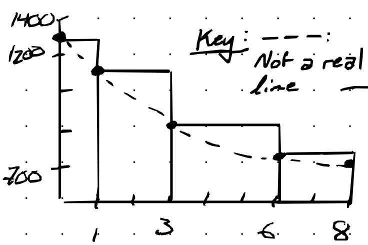
Pumped into tank (lt/hr); $R(t)$ = water removed, decreasing

@ $t=0$, 50,000 l in tank

(a) Estimate $R'(2)$.

$$R'(2) = \frac{R(3)-R(1)}{3-1} \rightarrow -120\ l/hr^2$$

(b) left RM Sum, 4 partitions; is over- or underestimate?




See the decreasing function

8050l, overestimate as $R(t)$ is decreasing on 0 to 8, and a left sum on a decreasing function yields an overestimation

$$\int_0^8 R(t)\,dt = (1-0)f(0) + (3-1)f(1) + (6-3)f(3) + (8-6)f(6)$$

$$= 1340 + 2(1190) + 3(950) + 2(740)$$

$$= 8050 \quad \text{(decreasing, overest.)}$$

# Chapter 4 Quiz Review

**Points to study:** ★ Integrating trig funcs.
↳ ★ Definite & indefinite

Yippee!

★ Integrals using "u" substitution
★ Findinding limits @ infinity of sums
★ Second fundamental theorem of Calc
★ Average & mean integrals

(10) $\int \frac{-\sec x \tan x}{\sqrt{\sec x}}\,dx \to u = \sec x \quad du = \sec x \tan x\,dx$

$-1\int \frac{1}{\sqrt{u}}\,du \to -1\int u^{-1/2}\,du \to -1\left[2u^{1/2}\right] + C \to \boxed{-2\sqrt{\sec x} + C}$

(12) $\int \frac{2x}{\sqrt{x+1}}\,dx \to \begin{matrix} u = x+1 \\ du = dx \end{matrix} \Rightarrow x = u-1$

$\to 2\int \frac{x}{\sqrt{x+1}}\,dx \to 2\int \frac{u-1}{\sqrt{u}}\,du \to 2\int \left(u^{1/2} - u^{-1/2}\right) du$

$\to 2\left(\frac{2u^{3/2}}{3} - 2u^{1/2}\right) + C \to \boxed{\frac{4(x+1)^{3/2}}{3} - 4(x+1)^{1/2} + C}$

(6) $s(n) = \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right)^2 \left(\frac{2}{n}\right)$ find lim. of $s(n)$ as $n \to \infty$

$\to s(n) = \sum_{i=1}^{n} f(m_i)\Delta x \to s(n) = \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right)^2 \left(\frac{2}{n}\right)$

$\to \frac{2}{n}\sum_{i=1}^{n}\left(1 + \frac{2i}{n} + \frac{i^2}{n^2}\right) \to \frac{2}{n}\left(\sum_{i=1}^{n} 1 + \frac{2}{n}\sum_{i=1}^{n} i + \frac{1}{n^2}\sum_{i=1}^{n} i^2\right)$

$= \frac{2}{n}\left(n \cdot \frac{6n}{6n} + \frac{2}{n}\left(\frac{n(n+1)}{2}\right) + \frac{1}{n^3}\left(\frac{\cancel{n}(n+1)(2n+1)}{6}\right)\right)$

$\to \frac{\cancel{2}}{n}\left(\frac{6n^2 + 6n^2 + 6n + 2n^2 + 3n + 1}{\cancel{3}6n}\right)$

$= \frac{14n^2 + 9n + 1}{3n^2} \to \lim_{n\to\infty} s(n) = \frac{14}{3}$

(4) $y' = 4x+4$ solve for $y=f(x)$

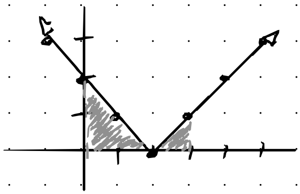
$$dx \cdot \frac{dy}{dx} = (4x+4)\,dx \to \int dy = \int (4x+4)\,dx$$

$$\to y = \frac{4x^2}{2} + 4x + C \to y = 2x^2+4x+C$$

$$\to 15 = 2(2^2)+4(2)+C$$

$$\to C = -1 \to \boxed{f(x) = 2x^2+4x-1}$$

(11) $\int_0^3 |x-2|\,dx$ $\to y = |x-2|$ vertex $=(2,0)$

$\to$ 

$$\frac{1}{2}(2)(2)+\frac{1}{2}(1)(1) = 2+\frac{1}{2} = \frac{5}{2}$$

$$\int_0^2(-x+2)\,dx + \int_2^3 (x-2)\,dx$$

$$\to \left[\frac{-x^2}{2}+2x\right]_0^2 + \left[\frac{x^2}{2}-2x\right]_2^3$$

$$\to [2-0]+\left[\left(\frac{9}{2}-6\right)-(-2)\right] = \frac{5}{2}$$

(8) Average value of $f(x)=\cos x$ on $[0,\frac{\pi}{4}]$

$$\to \frac{1}{\frac{\pi}{4}-0}\int \cos x\,dx \to \frac{4}{\pi}\left[\sin\frac{\pi}{4} - \sin 0\right] \to \frac{4}{\pi}\left[\frac{\sqrt{2}}{2}+0\right]$$

$$\to \frac{2}{\pi}\left(\frac{\sqrt{2}}{1}\right) \to \boxed{\frac{2\sqrt{2}}{\pi}}$$

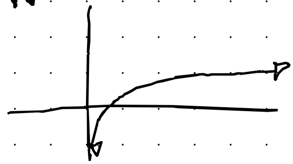
(3) $\int\left(\frac{x^2-x}{\sqrt{x}}\right)dx \to \frac{x^2}{x^{1/2}} - \frac{x}{x^{1/2}} \to x^{3/2}-x^{1/2}$

$$\int\left(x^{3/2}-x^{1/2}\right) \to \boxed{\frac{2x^{5/2}}{5} - \frac{2x^{3/2}}{3}+C}$$

(7) $\frac{d}{dx}\int_1^{2x^3}(t-3)^4\,dt \to \boxed{(2x^3-3)^4(6x^2)}$ FTC II

# 5.1 Natural Logs: Differentiation

$\ln x = (\log_e) x$



**Properties of ln:**

1. Domain is $(0,\infty)$ & range is $(-\infty,\infty)$
2. Function is increasing, continuous, & one to one
3. Concave downward

$y=\log_b x \rightarrow b^y = x$

$n=\log_m u \rightarrow m^n = u$

**More properties:**

1. $\ln 1 = 0$
2. $\ln(ab) = \ln a + \ln b \rightarrow$ because: $a^m \cdot a^n = a^{m+n}$
3. $\ln a^n = n \ln a$
4. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b \rightarrow$ because $\frac{a^m}{a^n} = a^{m-n}$

**Ex 1:**

a. $\ln\left(\frac{10}{9}\right) \rightarrow \ln(10) - \ln(9)$

b. $\ln\sqrt{3x+2} \rightarrow \ln(3x+2)^{1/2} \rightarrow \frac{1}{2}\ln(3x+2)$

c. $\ln\frac{6x}{5} \rightarrow \ln 6x - \ln 5 \rightarrow \ln 6 + \ln x - \ln 5$

d. $\ln\frac{(x^2+3)^2}{x(\sqrt[3]{x^2+1})} \rightarrow \ln(x^2+3)^2 - \ln\left(x(\sqrt[3]{x^2+1})\right)$

$\rightarrow 2\ln(x^2+3) - \left(\ln x + \frac{1}{3}\ln(x^2+1)\right)$

$\rightarrow 2\ln(x^2+3) - \ln x - \frac{1}{3}\ln(x^2+1)$

Ex 2:

a. $\ln 2 = 0.693$

b. $\ln 32 = 3.466$

c. $\ln 0.1 = -2.303$

Definition of ln functions

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0$$

Definition of e

$$\ln e = \int_1^e \frac{1}{t}\,dt = 1$$

Derivative

$$\frac{d}{dx}(\ln x) = \boxed{\frac{1}{x}}$$

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\cdot\frac{du}{dx} = \boxed{\frac{u'}{u}}$$

Integral

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \frac{1}{u}\,du = \ln|u| + C$$

---

Ex 3:

a. $\frac{d}{dx}[\ln(2x)] = \frac{2}{2x} \to \frac{1}{x}$

b. $\frac{d}{dx}[\ln(x^2+1)] = \frac{2x}{x^2+1}$

c. $\frac{d}{dx}[x\ln x] \to x\left(\frac{1}{x}\right) + 1\ln x \to \boxed{1+\ln x}$

d. $\frac{d}{dx}[(\ln x)^3] \to 3(\ln x)^2\left(\frac{1}{x}\right) = \boxed{\frac{3(\ln x)^2}{x}}$ or $\frac{3}{x}(\ln x)^2$

Ex 4: find $f'(x)$

$f(x) = \ln\sqrt{x+1}$

$\rightarrow \frac{1}{2}\ln(x+1) \rightarrow f'(x) = \frac{1}{2} \cdot \frac{1}{(x+1)} = \boxed{\frac{1}{2(x+1)}}$

Ex 5:

$f(x) = \ln \frac{x(x^2+1)^2}{\sqrt{2x^3-1}} \rightarrow \ln(x(x^2+1)^2) - \frac{1}{2}\ln(2x^3-1)$

$\rightarrow f'(x) = \frac{1}{x} + 2\left(\frac{2x}{x^2+1}\right) - \frac{1}{2}\left(\frac{6x^2}{2x^3-1}\right)$

$\rightarrow \boxed{\frac{1}{x} + \frac{4x}{x^2+1} - \frac{3x^2}{2x^3-1}}$

5.2

Warm up: 1) $y = (3x)\ln(x^2-2)$

$\rightarrow (3x)\left(\frac{2x}{x^2-2}\right) + (3)(\ln(x^2-2))$

$\rightarrow \frac{6x^2}{x^2-2} + \ln(x^2-2)^3$

2) $g(x) = \ln\left(\frac{(1+x^2)^2}{\sqrt{4x-1}}\right) - \cancel{x}2\ln(1+x^2) - \left(\frac{1}{2}\ln(4x-1)\right)$

$\rightarrow 2\left(\frac{2x}{1+x^2}\right) - \frac{1}{2}\left(\frac{4}{4x-1}\right) \rightarrow \frac{4x}{1+x^2} - \frac{4}{8x-2}$

$\rightarrow \frac{4x}{1+x^2} - \frac{2}{4x-1} \rightarrow \underbrace{\frac{16x^2-4x}{4x-1+4x^3-x^2} - \frac{2+2x^2}{4x-1+4x^3-x^2}}$

$\rightarrow \frac{14x^2-4x-2}{4x^3-x^2+4x-1} = g'(x)\} \longrightarrow$ My special Brain

# 5.2 Natural Log Integration

Use log or ln differentiation when base & exponent are both a function of "x"

Eg.: $\frac{dy}{dx}\left(y = x^{x-1}\right) \rightarrow$

On next test

Steps:

1. Take ln of both sides — $y = x^{x-1}$, $\ln y = \ln x^{x-1}$
2. Use a property to simplify the right side — $\ln y = (x-1)\ln x$
3. Differentiate both sides with respect to "x" — $\frac{d}{dx}(\ln y) = \frac{d}{dx}\left((x-1)\ln x\right)$

$$y \cdot \frac{y'}{y} = \left[(x-1)\left(\frac{1}{x}\right) + (1)(\ln x)\right] y$$

4. Isolate $y'$: Multiply both sides by $y$ to get $y'$ by itself — $y' = y\left(\frac{x-1}{x} + \ln x\right)$
5. Substitute $y$ with given — $y' = x^{x-1}\left(\frac{x-1}{x} + \ln x\right)$

## Theorem 5.5: Log Rule for Integration

Let $u$ be a differentiable function of "x."

1. $\int \frac{1}{x}\,dx = \ln|x| + C$
2. $\int \frac{1}{u}\,du = \ln|u| + C$

Always add abs() ↳ Just in case

Ex 1: $\int \frac{2}{x}\,dx \rightarrow 2\int \frac{1}{x}\,dx = 2(\ln|x|) + C$

Ex 2: $\int \frac{1}{4x-1}\,dx(4)$ $u = 4x-1$ $du = 4\,dx$

$\Rightarrow \frac{1}{4}\left(\int \frac{1}{u}\,du\right) \rightarrow \frac{1}{4}\ln|u| + C$

Ex 3: Find the area of $y=\frac{x}{x^2+1}$, bounded by $x$-axis & $x=3$

$\frac{1}{2}\int_0^3 \frac{x}{x^2+1}\,dx\,(2)$ $\quad u=x^2+1 \quad du=2x\,dx$ $\quad x=0,\ u=1;\ x=3,\ u=10$

$\to \frac{1}{2}\int_1^{10}\frac{1}{u}\,du \to \frac{1}{2}\left[\ln|u|\right]_1^{10}$

$\to \frac{1}{2}\left[\ln 10-\ln 1\right] \to \left(\frac{1}{2}\ln(10)\right)=\text{Area}$

Ex 4: (a) $\int \frac{3x^2+1}{x^3+x}\,dx \to u=x^3+x \quad du=(3x^2+1)\,dx$

$\to \int \frac{1}{u}\,du \to \ln|u|+c \to \ln|x^3+x|+c$

(b) $\int \frac{\sec^2 x}{\tan x}\,dx \quad u=\tan x \quad du=\sec^2 x\,dx$

$\to \int \frac{1}{u}\,du \to \ln|u|+c \to \ln|\tan x|+c$

(c) $\frac{1}{2}\int \frac{x+1}{x^2+2x}\,dx\cdot 2 \quad u=x^2+2x \quad du=(2x+2)\,dx$

$\to \frac{1}{2}\int \frac{1}{u}\,du \to \frac{1}{2}\ln|u|+c \to \frac{1}{2}\ln|x^2+2x|+c$

(d) $\int \frac{1}{3x+2}\,dx \quad u=3x+2 \quad du=3\,dx$

$\to \frac{1}{3}\int \frac{1}{u}\,du \to \frac{1}{3}\ln|u|+c \to \frac{1}{3}\ln|3x+2|+c$

(e) $\int \frac{x^2+x+1}{x^2+1}\,dx$ long division

exponents are equal!

↓

DIVIDE

$\to x^2+1\overline{)x^2+x+1}$ $\quad 1+\frac{x}{x^2+1} \to \int\left(1+\frac{x}{x^2+1}\right)dx$

$1\,(x^2+1)\ \ \cancel{x^2}\ \ 1$

$+x$ remainder

$\to \int 1\,dx + \int \frac{x}{x^2+1}\,dx \qquad u = x^2+1 \quad du = 2x\,dx$

$\to x + \frac{1}{2}\int \frac{1}{u}\,du \to x + \frac{1}{2}\ln|u| + C$

$\to \boxed{x + \frac{1}{2}\ln|x^2+1| + C}$

(f) $\int \frac{2x}{(x+1)^2}\,dx \qquad u = x+1 \quad du = 1\,dx$

$x = u-1$

$\to 2\int \frac{u-1}{u^2}\,du \to 2\int \left(\frac{1}{u} - \frac{1}{u^2}\right)du$

$\to 2\int \left(\frac{1}{u} - u^{-2}\right)du \to 2\left(\ln|u| - (-u^{-1})\right) + C$

$\to \boxed{2\left(\ln|x+1| + \frac{1}{u}\right) + C}$

(g) $\left(\frac{dy}{dx} = \frac{1}{x\ln x}\right) dx \to \int dy = \int \frac{1}{x\ln x}\,dx$

$u = \ln x$
$du = \frac{1}{x}\,dx$

$\to y = \int \frac{1}{u}\,du \to y = \ln|u| + C$

$\to \boxed{y = \ln|\ln|x|| + C}$

(h) $\int \tan x\,dx \to (-1)\int \frac{\sin x\,(-1)}{\cos x}\,dx \qquad u = \cos x\,dx$
$du = -\sin\,dx$

$\to -\int \frac{1}{u}\,du \to -\ln|u| + C \to \boxed{-\ln|\cos x| + C}$

$\to \ln|\cos x|^{-1} + C \to \ln\left|\frac{1}{\cos x}\right| + C \to \boxed{\ln|\sec x| + C}$

# Integrals of Trig. Funcs.

$\int \sin u \, du = -\cos u + c$ $\quad \int \cos u \, du = \sin u + c$

$\int \tan u \, du = -\ln|\cos u| + c$ $\quad \int \cot u \, du = \ln|\sin u| + c$

$\int \sec u \, du = \ln|\sec u + \tan u| + c$ $\quad \int \csc u \, du = -\ln|\csc u + \cot u| + c$

Ex 10: $\int_0^{\pi/4} \sqrt{1+\tan^2 x} \, dx$

$\rightarrow \int_0^{\pi/4} \sqrt{\sec^2 x} \, dx \rightarrow [\ln|\sec x + \tan x|]_0^{\pi/4}$

$\rightarrow [(\ln|\sec \frac{\pi}{4} + \tan \frac{\pi}{4}|) - (\ln|\sec 0 + \tan 0|)]$

$\rightarrow [(\ln(\sqrt{2}+1)) - (\ln(1+0))] = \boxed{\ln|\sqrt{2}+1|}$

---

on 43, remove $x$ from "9" in denominator

5.3 Warm-up

a) $\int \frac{4x^2}{x^3-7} dx \quad u = x^3 - 7 \quad du = 3x^2 \, dx$

$\rightarrow \frac{4}{3} \int \frac{1}{u} du \rightarrow \boxed{\frac{4}{3} \ln|x^3-7| + c}$

b) $\frac{1}{5}\int \csc(5x) \, dx \rightarrow u = 5x \quad du = 5 \, dx$

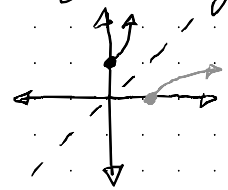
$\ln|\csc(5x) + \cot 5x| + c$

$\boxed{-\frac{1}{5} \ln|\csc(5x) + \cot(5x)| + c}$

# 5.3 Inverse Functions

## Facts to remember

- Switch $x$ & $y$ of function $\begin{cases} f(x) \Rightarrow (3,2) \\ f^{-1}(x) \Rightarrow (2,3) \end{cases}$
- Graph is therefore reflected over $y=x$ (the origin)



- To verify inverses:

and $\left[\begin{array}{l} f(f^{-1}(x)) = x \\ f^{-1}(f(x)) = x \end{array}\right.$ $\quad f \circ g \quad g \circ f$ must do both

- How to find the inverse function

- Switch $x$ & $y$ - Solve for $y$

Ex 1: Prove inverses

$f(x) = 2x^3 - 1 \qquad f^{-1}(x) = g(x) = \sqrt[3]{\frac{x+1}{2}}$

$\Rightarrow 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1 \Rightarrow 2\left(\frac{x+1}{2}\right) - 1 \Rightarrow x \checkmark$

$\Rightarrow \sqrt[3]{\frac{(2x^3-1)+1}{2}} - 1 \Rightarrow \sqrt[3]{\frac{2x^3}{2}} - 1 \Rightarrow \sqrt[3]{x^3} \Rightarrow x \checkmark$

$\therefore$ $f$ & $g$ are inverses of each other

## The existence of an Inverse function

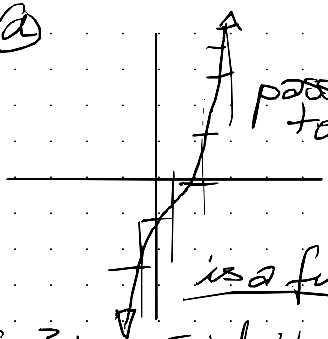
The inverse is only a function if:

ⓐ Must be "1 to 1", only 1 $x$ per $y$ & 1 $y$ per $x$

ⓑ Strictly increasing / decreasing

Ex 2: $f(x) = x^3 + x - 1$ $\qquad$ $f(x) = x^3 - x + 1$

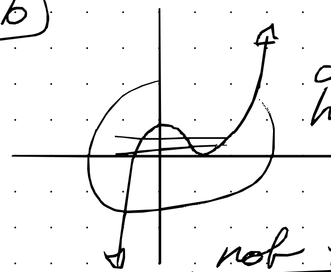
(a)



passes both tests

is a func.

(b)



doesn't pass horizontal line test

not a func.

Ex 3: Find the inverse

$f(x) = \sqrt{2x-3} \rightarrow x = \sqrt{2y-3} \rightarrow x^2 = 2y - 3$

$\rightarrow x^2 + 3 = 2y \rightarrow y = \frac{x^2+3}{2}$

$\rightarrow f^{-1}(x) = \frac{1}{2}x^2 + \frac{3}{2}, \quad x \geq 0$

## Properties of inverses

1. If "f" is continuous, "$f^{-1}$" is also continuous
2. If $f$ is increasing, $f^{-1}$ is also increasing
3. " decreasing, " decreasing
4. If $f$ is diff'able on interval containing "c" & $f'(c) \neq 0$, then $f^{-1}$ is diff'able at $f(c)$.

## Derivative of inverse function

Let $f$ be diff'able function, inverse $(g)$ is diff'able at any $x$ where $f'(g(x)) \neq 0$. Moreover

$$g'(x) = \frac{1}{f'(g(x))} \qquad f'(g(x)) \neq 0$$

Ex 5: $f(x)=\frac{1}{4}x^3+x-1$

ⓐ What is $f^{-1}(x)$ @ $x=3$

ⓑ What is $(f^{-1})'(x)$ @ $x=3$
(find slope of $f^{-1}(x)$ @ $x=3$)

$$(f^{-1})'(3)=\frac{1}{f'(f^{-1}(3))}$$

Remember switched $x$ & $y$!

| | $x$ | $y$ |
|---|---|---|
| $f$ | ? 2 | 3 |
| $f^{-1}$ | 3 | ? 2 |

$$3=\frac{1}{4}x^3+x-1 \to \underset{q}{\frac{1}{4}}x^3+x\underset{p}{-4}=0$$

ⓐ $\boxed{2}$

$$\frac{p}{q}=\frac{\pm1,\pm2,\pm4}{\pm1,\pm\frac{1}{2},\pm\frac{1}{4}}=\pm1,\pm2,\pm4$$

~~1 | 1/4 0 1 −4~~

2 | $\frac{1}{4}$ 0 1 −4
$\quad\ \ \frac{1}{2}$ 1 4
$\quad\frac{1}{4}$ $\frac{1}{2}$ 2 0

ⓑ $$(f^{-1})'(3)=\frac{1}{f'(f^{-1}(x))}\to\frac{1}{f'(2)}\to\frac{1}{\frac{3}{4}(4)+1}$$

$\to \boxed{\frac{1}{4}}$ slope of $f^{-1}$ @ $x=3$

Ex 6: $f(x)=x^3$ for $(x\geq 0)$, $f^{-1}(x)=\sqrt{x}$

show that slopes of $f(x)$ & $f^{-1}(x)$ are reciprocals @ ⓑ $(3,9)$ and $(9,3)$

$f(x)\to(3,9)$ $\quad f'(3)=$

$f^{-1}(x)\to(9,3)$ $\quad (f^{-1})'(9)=$

$$(f^{-1})'(x)=\frac{1}{2\sqrt{x}}$$

$$(f^{-1})'(9)=\frac{1}{2\sqrt{9}}=\boxed{\frac{1}{6}}$$

$\boxed{6 \text{ \& } \frac{1}{6} \text{ are reciprocals}}$

$f'(x)=2x$

$f'(3)=2(3)=6$ $\to$ $f(x)=x^2$ $\quad y=\sqrt{x}$ $\quad x\geq 0$

$x=y^2$

$f^{-1}(x)=\sqrt{x}$

$f^{-1}(9)=\boxed{3}$

# 5.4 Differentiation & Integration of Exponential Funcs.

Warm-up: $f(x) = 4x^3 + 3x - 4$, $a = 3$ → Slope of inverse at $x = 3$

$\to f'(x) = 12x^2 + 3$ $\quad f(1) = 3 \to 3 = 4x^3 + 3x - 4$

$\to 0 = 4x^3 + 3x - 7$

$\to (f^{-1})'(3) = \dfrac{1}{f'(f^{-1}(3))} \to f^{-1}(3) = 1$

$\to \dfrac{1}{f'(1)} \to \dfrac{1}{12(1)^2 + 3} \to \dfrac{1}{12+3} \to \boxed{\dfrac{1}{15}}$

| | x | y |
|---|---|---|
| f | 1 | 3 |
| $f^{-1}$ | 3 | 1 |

$3 = 4x^3 + 3x - 4 \to 0 = 4x^3 + 3x - 7$ (q = 4, p = 7)

$\dfrac{p}{q} = \dfrac{\pm 1, \pm 7}{\pm 1, \pm 2, \pm 4} = \pm 1, \pm \frac{1}{2}, \pm \frac{1}{4}, \pm 7, \pm \frac{7}{2}, \pm \frac{7}{4}$ } Must use p's & q's

$$\begin{array}{c|cccc} 1 & 4 & 0 & 3 & -7 \\ & \downarrow & 4 & 4 & 7 \\ \hline & 4 & 4 & 7 & 0 \end{array} \to 4x^2 + 4x + 7$$

---

$f(x) = e^x$ $\quad f'(x) = ?$ $\quad \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$h = .001$ $\quad \to \boxed{f'(x) = e^x}$ $\quad = \lim\limits_{h \to 0} \dfrac{((e^{x+h}) - (e^x))}{h}$

$y_1 = e^x$

$y_2 = ((e^{x+.001}) - (e^x))/.001$

---

Inverse of "ln" func. is called natural exponential func, it's denoted by $f^{-1}(x) = e^x$. $y = e^x$ iff $x = \ln y$

---

Operations with exponential functions

1. $e^a \cdot e^b = e^{(a+b)}$
2. $e^a / e^b = e^{(a-b)}$

---

**Ex1:** $7 = e^{x+1} \to \ln 7 = \ln(e^{x+1})$

$\to \ln 7 = (x+1)\ln e \to \ln 7 = (x+1)(1) = x+1$

$\to (\ln 7) - 1 = x$

Ex 2: Solve

$\ln(2x-3)=5 \rightarrow 2x-3=e^5$

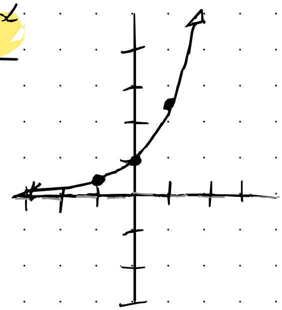
$\Rightarrow x = \frac{(e^5+3)}{2}$

## Sketching $f(x)=e^x$

| $x$ | $f(x)$ |
|---|---|
| $-1$ | $e^{-1}$ |
| $0$ | $1$ |
| $1$ | $e^1$ |

$\rightarrow e^{-1} \approx .4$

$\rightarrow e \approx 2.7$



## Properties of e functions

1. domain $(-\infty,\infty)$, range $(0,\infty)$
2. Continuous, increasing, one-to-one
3. Concave up
4. $\lim_{x\to\infty} e^x = \infty$, $\lim_{x\to -\infty} e^x = 0$

## Derivative of e functions

1. $\frac{d}{dx}[e^x]=e^x$
2. $\frac{d}{dx}[e^u]=e^u\frac{du}{dx}=e^u\cdot u'$

Ex 3: ⓐ

$y=e^{2x-1} \Rightarrow y'=e^{2x-1}\frac{du}{dx} \rightarrow e^{2x-1}\cdot 2 \Rightarrow \boxed{2e^{2x-1}}$

ⓑ $y=e^{-\frac{3}{x}} \Rightarrow u=-3x^{-1}$ $u'=\frac{3}{x^2}$

$\Rightarrow e^{-3/x}\cdot\frac{3}{x^2} \Rightarrow \boxed{\frac{3e^{-3/x}}{x^2}}$

Ex 4: $f(x) = xe^{x} \rightarrow \boxed{f'(x) = xe^{x} + 1e^{x}}$

$\rightarrow 0 = e^{x}(x+1) \rightarrow e^{x} \neq 0, \quad x+1 = 0 \rightarrow x = -1$

Testing $\rightarrow D: (-\infty, \infty) \rightarrow (-\infty, -1)(-1, \infty)$

$-2 \qquad 0$

$\rightarrow -2e^{-2} + e^{-2} \rightarrow \frac{-2}{e^{2}} + \frac{1}{e^{2}} \rightarrow (-)$

$\rightarrow 0e^{0} + 0^{e} \rightarrow 0 \rightarrow +$

Relative min. is @ $\frac{-1}{e}$ when $x = -1$.

$f(-1) = -1e^{-1}$

## Integration of e funcs.

1. $\int e^{x}\,dx = e^{x} + C$
2. $\int e^{u}\,du = e^{u} + C$

Ex 4: $\int e^{3x+1}\,dx \rightarrow \quad u = 3x+1, \ du = 3dx$

$\frac{1}{3}\int e^{u}\,du \rightarrow \frac{1}{3}[e^{u}] + C \rightarrow \boxed{\frac{1}{3}e^{3x+1} + C}$

Ex 5:

$\int 5xe^{-x^{2}}\,dx \qquad u = -x^{2} \quad du = -2x\,dx$

$\frac{5}{-2}\int xe^{-x^{2}}\,dx^{-2} \rightarrow -\frac{5}{2}\int e^{u}\,du \rightarrow -\frac{5}{2}(e^{-x^{2}}) + C$

Ex 6: (a) $\int \frac{e^{1/x}}{x^{2}}\,dx \rightarrow u = 1/x \quad du = \frac{-1}{x^{2}}\,dx$

$\rightarrow -\int e^{u}\,du \rightarrow \boxed{-e^{1/x} + C}$

# 5.5 Bases other than e

Warm-up:

① $y=(4x^5+3)e^{4x^4}$ $\rightarrow$ $\boxed{20x^4(e^{4x^4})+(4x^5+3)(e^{4x^4}(16x^3))}$

② $\int 60x^3 e^{3x^4-2}\,dx$  $u=3x^4-2$  $du=12x^3\,dx$

$\rightarrow 5\int e^u\,du \rightarrow 5[e^u]+C \rightarrow \boxed{5e^{3x^4-2}+C}$

③ $\int -20\csc^2 4x \cdot e^{\cot 4x}\,dx$  $u=\cot 4x$  $du=-4\csc^2(4x)$

$5\int -4\csc^2 4x \cdot e^u \rightarrow 5\int e^u\,du \rightarrow$ $\boxed{5e^{\cot 4x}+C}$

## Exponential Function to Base "a"

If "a" is positive, $(a\neq 1)$ and $x$ is real, exponential function to the base "a" is $a^x$

$a^x = e^{(\ln a)x}$  $b^{(\log_b(u))} = u$

## Change of base formula

$\log_b a = \dfrac{\log_{\text{desired base}} a}{\log_{\text{desired base}} b}$

desired base = e or 10

b = starting base

E.g.: $\log_3 4 = \dfrac{\ln 4}{\ln 3} = \dfrac{\log_{10} 4}{\log_{10} 3} = \dfrac{\log_{13} 4}{\log_{13} 3}$

Ex 1: Radioactive half-life

model: $y = a \cdot \left(\frac{1}{2}\right)^{t/\text{half life}}$ (a: sample; t: years)

$\rightarrow 1\cdot\left(\frac{1}{2}\right)^{\left(\frac{10000}{5715}\right)} \rightarrow 0.297\,g$

1g carbon-14 after 10,000 years

## Log function to base "a"

$\log_a x = \dfrac{1}{\ln a}\ln x$  $\log_a x = \dfrac{\ln x}{\ln a} = \dfrac{1}{\ln a}\cdot \ln x$

## Properties of Logs

1. $\log_a 1 = 0$
2. $\log_a xy = \log_a x + \log_a y$
3. $\log_a x^n = n \log_a x$
4. $\log_a \frac{x}{y} = \log_a x - \log_a y$

Memorize

## Properties of inverse functions

1. $y = a^x$ iff $x = \log_a y$
2. $a^{\log_a x} = x$, for $x > 0$
3. $\log_a a^x = x$

Ex 2:

(a) $3^x = \frac{1}{81} \to 3^x = \frac{1}{3^4} \to 3^x = 3^{-4} \to x = -4$

(b) $\log_2 x = -4 \to 2^{-4} = x \to \frac{1}{16} = x$

$\frac{du}{dx} = u'$

## Derivatives for bases other than e

1. $\frac{d}{dx}[a^x] = (\ln a)a^x$
2. $\frac{d}{dx}[a^u] = (\ln a)a^u \frac{du}{dx}$
3. $\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)x}$
4. $\frac{d}{dx}[\log_a u] = \frac{1}{(\ln a)u} \frac{du}{dx} = \frac{u'}{u \ln a}$

Ex 3: Derive

(a) $y = 2^x$ $\quad a = 2 \to \boxed{\frac{dy}{dx} = (\ln 2)2^x}$

(b) $y = 2^{3x}$ $\quad u = 3x \quad u' = 3 \to \boxed{\frac{dy}{dx} = (\ln 2)(2^{3x})(3)}$

ⓒ $y = \log_{10} \cos x \rightarrow \frac{u'}{u \ln a}$

$\frac{dy}{dx} = \frac{-\sin x}{\cos x \ln 10} \rightarrow \boxed{\frac{-\tan x}{\ln 10}}$

ⓓ $y = \log_3 \frac{\sqrt{x}}{x+5} \rightarrow \frac{1}{2}\log_3 x - \log_3 (x+5)$

$\rightarrow y' = \left(\frac{1}{2}\left(\frac{1}{x \ln 3}\right)\right) - \frac{1}{(x+5)\ln 3} \rightarrow \boxed{\frac{1}{2 \ln 3\, x} - \frac{1}{(x+5)\ln 3}}$

<u>Integrals of Bases other than "e"</u>

$\int a^x \, dx = \left(\frac{1}{\ln(a)}\right) a^x + C \rightarrow \int a^u \, du = \frac{1}{\ln a} a^u + C$

<u>Ex 4:</u> ⓐ $\int 2^x \, dx \rightarrow \left(\frac{1}{\ln 2}\right) 2^x + C = \boxed{\frac{2^x}{\ln 2} + C}$

ⓑ $\int 3^{4x} dx \quad u = 4x \quad du = 4\, dx$

$\rightarrow \frac{1}{4} \int 3^u \, du \rightarrow \frac{1}{4} \frac{1}{\ln 3} (3^u) + C$

$\rightarrow \frac{1}{4} \frac{1}{\ln 3} (3^{4x}) + C \rightarrow \boxed{\frac{3^{4x}}{4 \ln 3} + C}$

ⓒ $\int \sin x \,(2^{\cos x})\, dx \qquad u = \cos x \quad du = -\sin x \, dx$

$-\int 2^u \, du \rightarrow -\left(\frac{1}{\ln 2}\right)\left(2^{\cos x}\right) + C \rightarrow \boxed{\frac{-2^{\cos x}}{\ln 2} + C}$

<u>Ex 5:</u>

ⓐ $\frac{d}{dx}[e^e] = 0$ ⓑ $\frac{d}{dx}[e^x] = e^x$ ⓒ $\frac{d}{dx}[x^e] = ex^{(e-1)}$

ⓓ $\frac{d}{dx}[x^x] \Rightarrow \ln y = \ln x^x \rightarrow \frac{y'}{y} = x \ln x \rightarrow \frac{y'}{y} = x\left(\frac{1}{x}\right) + 1 \ln x$

$\rightarrow y' = y(1 + \ln x) \rightarrow y' = x^x (1 + \ln x)$

Compounded $n$ times/year: $A = P\left(1+\frac{r}{n}\right)^{nt}$

Compounded continuously: $A = Pe^{rt}$

<u>Ex 6:</u> $n=4$ $P=2500$ 5%

$n=12$

$n=4$ $A = 2500\left(1+\frac{0.05}{4}\right)^{4(5)} = 3205.09$

$n=12$ $A = 2500\left(1+\frac{0.05}{12}\right)^{(12(5))} = 3208.40$

continuously: $A = 2500e^{0.05(5)} = 3210.06$

# 5.6 Inverse Trig Functions

Warm-up: (1) $y = x^3 3^{2x}$ $\quad 3x^2(u') + (x^3)(u)$

$\rightarrow u' = (\ln 3)\, 3^{2x}(2) \rightarrow 3^{2x}(\ln 3)(2)$

$\rightarrow 3x^2\left(3^{2x}(\ln 3)2\right) + x^3\left(3^{2x}\right)$

$\rightarrow 3^{2x}x^2\left(6(\ln 3) + x\right)$

(2.) $f(x) = \log_5(3x+4) \rightarrow \dfrac{3}{(3x+4)\ln 5} \rightarrow \dfrac{3}{\ln 5\,(3x+4)}$

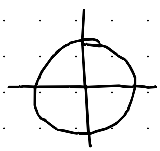
---

## Inverse Trig Functions

| | | D | R |
|---|---|---|---|
| $y = \sin^{-1} x$ | iff $\sin y = x$ | $[-1, 1]$ | $[-\frac{\pi}{2}, \frac{\pi}{2}]$ |
| $y = \cos^{-1} x$ | iff $\cos y = x$ | $[-1, 1]$ | $[0, \pi]$ |
| $y = \tan^{-1} x$ | iff $\tan y = x$ | $(-\infty, \infty)$ | $(-\frac{\pi}{2}, \frac{\pi}{2})$ |

Ex 1:

(a) $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$



(b) $\cos^{-1}(0) = \frac{\pi}{2}$

(c) $\tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right) \frac{y}{x} \rightarrow \frac{\pi}{3}$

$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

(d) $\sin^{-1}(0.3) \rightarrow 0.305$

Ex 2:

$\overset{\tan}{(}\arctan(2x-3)) = \overset{\tan}{\left(\frac{\pi}{4}\right)} \rightarrow 2x - 3 = \tan\left(\frac{\pi}{4}\right) \rightarrow 2x - 3 = 1 \rightarrow x = 2$

Ex 3: (a) $\cos(\cos^{-1}(\frac{1}{5})) \rightarrow \frac{1}{5}$ $\qquad$ (e) $\tan^{-1}(\tan \frac{4\pi}{3}) \rightarrow \frac{\pi}{3}$

(b) $\tan(\tan^{-1}(5.2)) \rightarrow 5.2$ $\qquad$ (d) $\sin^{-1}(\sin \frac{5\pi}{6}) \rightarrow \frac{\pi}{6}$

(c) $\sin(\sin^{-1}(3/2)) \rightarrow$ undefined

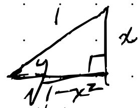
(d) $\cos^{-1}(\cos \frac{\pi}{4}) \rightarrow \frac{\pi}{4}$

sin → reflect y-axis
cos → reflect x-axis
tan → reflect origin

Ex 4:

ⓐ $y=\sin^{-1} x,\ 0 < y < \frac{\pi}{2}$ find $\cos y$

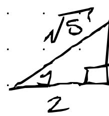
$\rightarrow \sin^{-1} \frac{x^{\text{opp}}}{1_{\text{hyp}}}$




$\cos y = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$

ⓑ $y = \text{arcsec}\left(\frac{\sqrt{5}^{\text{Hyp}}}{2_{\text{Adj}}}\right)$ find $\tan y$

$\sec\theta = \frac{\text{Hyp}}{\text{Adj}}$ $\cos\theta = \frac{\text{Adj}}{\text{Hyp}}$




$\sqrt{5-4} \rightarrow \sqrt{1} \rightarrow 1$

$\tan y \rightarrow \frac{1}{2}$

Derivatives of Inverse Trig funcs.

"co-"'s just negative

$$\frac{d}{dx}\left[\sin^{-1} u\right] = \frac{u'}{\sqrt{1-u^2}} \rightarrow \frac{d}{dx}\left[\cos^{-1} u\right] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}\left[\tan^{-1} u\right] = \frac{u'}{1+u^2} \rightarrow \frac{d}{dx}\left[\cot^{-1} u\right] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}\left[\sec^{-1} u\right] = \frac{u'}{|u|\sqrt{u^2-1}} \rightarrow \frac{d}{dx}\left[\csc^{-1} u\right] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

---

Ex 4: ⓐ $\sin^{-1}(2x) \rightarrow \frac{2}{\sqrt{1-(2x)^2}} \rightarrow \frac{2}{\sqrt{1-4x^2}}$

ⓑ $\tan^{-1}(3x) \rightarrow \frac{3}{1+(3x)^2} \rightarrow \frac{3}{1+9x^2}$

ⓒ $\sin^{-1}(\sqrt{x}) \rightarrow \frac{\frac{1}{2\sqrt{x}}}{\sqrt{1-x}} \rightarrow \frac{1}{2\sqrt{x}} \cdot \frac{1}{\sqrt{1-x}} \rightarrow \frac{1}{2\sqrt{x-x^2}}$

ⓓ $\sec^{-1}(e^{2x}) \rightarrow \frac{2e^{2x}}{|e^{2x}|\sqrt{(e^{2x})^2-1}} \rightarrow \frac{2e^{2x}}{e^{2x}\sqrt{e^{4x}-1}} \rightarrow \frac{2}{\sqrt{e^{4x}-1}}$

Ex 5:

$y = \sin^{-1} x + x\sqrt{1-x^2}$

$\rightarrow \frac{1}{\sqrt{1-x^2}} + x\left(\frac{1}{2}(1-x^2)^{-1/2}(-2x)\right) + 1(1-x^2)^{1/2}$

$\rightarrow \frac{1}{\sqrt{1-x^2}} + \frac{-x^2}{\sqrt{1-x^2}} + \frac{\sqrt{1-x^2}}{1}\frac{(\sqrt{1-x^2})}{(\sqrt{1-x^2})}$

$\rightarrow \frac{1-x^2+1-x^2}{\sqrt{1-x^2}} \rightarrow 2-2x^2 \rightarrow \frac{2(1-x^2)}{\sqrt{1-x^2}} \rightarrow 2\sqrt{1-x^2}$

# 5.7 Inverse Trig. Function Integration

Warm up: $12x^2$

1. $y = \sec^{-1} 4x^3 \rightarrow \frac{1}{|u|\sqrt{u^2-1}} \rightarrow \frac{12x^2}{|4x^3|\sqrt{16x^6-1}}$

2. $y = \cot^{-1} 2x^4$ $\quad 8x^3 \quad \frac{-8x^3}{4x^8+1}$

3. $y = \cos^{-1} 2x^2$ $\quad 4x \quad \frac{-4x}{\sqrt{1-4x^4}}$

---

## Integrals Involving Inverse Trig. Functions

1. $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin \frac{u}{a} + C$

2. $\int \frac{du}{a^2+u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$

3. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C$

"co-"s are these formulae but negative

---

Ex 1:

$u = x \quad du = 1\,dx \quad a = 2$

(a) $\int \frac{1}{\sqrt{4-x^2}}\,dx$ arcsin?

$\rightarrow \boxed{\arcsin \frac{x}{2} + C}$

$u = 3x \quad du = 3\,dx \quad a = \sqrt{2}$

(b) $\frac{1}{3}\int \frac{1\,(3)}{2+9x^2}\,dx$ arctan?

$\frac{1}{3}\left(\frac{1}{\sqrt{2}} \arctan \frac{3x}{\sqrt{2}} + C\right) \rightarrow \frac{1}{3\sqrt{2}} \arctan \frac{3x}{\sqrt{2}} + C$

$u = 2x \quad du = 2\,dx \quad a = 3$

(c) $\int \frac{1\,(2)}{(2)x\sqrt{4x^2-9}}$ arcsec?

$= \frac{1}{3} \operatorname{arcsec} \frac{|2x|}{3} + C$

Ex 2:

$\int \frac{1}{(e^x)\sqrt{e^{2x}-1}}(e^x)$ $u=e^x$ $du=e^x\,dx$ $a=1$

arcsec? $\frac{1}{u\sqrt{u^2-a^2}}\,du$

$\rightarrow \frac{1}{1}\text{arcsec}\frac{|e^x|}{1}+C \rightarrow \text{arcsec}|e^x|+C$

Ex 3: $\int \frac{x+2}{\sqrt{4-x^2}}\,dx$ arc sin $u=x$ $du=dx$ $a=2$

$\rightarrow \int \frac{x}{\sqrt{4-x^2}}\,dx + 2\int \frac{1}{\sqrt{4-x^2}}\,dx$ (u-sub) (arcsin)

$\rightarrow u=4-x^2$ $du=-2x\,dx$

$-\frac{1}{2}\int \frac{1}{u^{1/2}}\,du \rightarrow -\frac{1}{2}\int u^{-1/2}\,du \rightarrow -\frac{1}{2}\left[\frac{2u^{1/2}}{1}\right]+C$

$\rightarrow -\sqrt{4-x^2}+C$

$\rightarrow u=x$ $du=dx$ $a=2$ coefficient $=2$

$+2\left(\arcsin\frac{x}{2}\right)+C$

$\rightarrow -\sqrt{4-x^2}+2\arcsin\frac{x}{2}+C$

Ex 4: $\int \frac{1}{x^2-4x+7}\,dx$ arctan?

Use "completing the square"

$x^2-4x+7 \rightarrow (x^2-4x+\underline{4})+7-\underline{4}$ $\left(\frac{b}{2}\right)^2$

$\rightarrow (x-2)^2+3$

$u^2+a^2$ $u=x-2$ $du=dx$ $a=\sqrt{3}$

$\rightarrow \int \frac{1}{(x-2)^2+3}\,dx$ arctan

$\frac{1}{\sqrt{3}}\arctan\frac{x-2}{\sqrt{3}}+C$

Ex 5: $\displaystyle\int_{3/2}^{9/4} \frac{1}{\sqrt{3x-x^2}}$ arcsin

Complete the Square $\left(\frac{b}{2}\right)^2$ $\left(-\frac{3}{2}\right)^2$

$\rightarrow \displaystyle\int \frac{1}{\sqrt{9/4-(x-\frac{3}{2})^2}}\,dx$

$3x-x^2$

$\rightarrow -x^2+3x$

$-\left(x^2-3x+\frac{9}{4}\right)+\frac{9}{4}$

$= \arcsin \dfrac{x-3/2}{3/2} + C$

$\rightarrow \left[\arcsin \dfrac{2x-3}{3} + C\right]$ indef.

$\frac{9}{4}-\left(x-\frac{3}{2}\right)^2$

$a^2 - u^2 \rightarrow \arcsin$

$\rightarrow \left[\arcsin \dfrac{2x-3}{3}\right]_{3/2}^{9/4} = \left[\arcsin \frac{1}{2} - \arcsin 0\right]$

$\left(\frac{9}{4}-\frac{3}{2}\right)\frac{2}{3} \rightarrow \left(\frac{3}{4}\right)\frac{2}{3} \rightarrow \frac{6}{12} \rightarrow \frac{1}{2}$

$\rightarrow \left[\frac{\pi}{6} - 0\right] \rightarrow \frac{\pi}{6}$

# 8.7 Indeterminate Forms & L'Hôpital's Rule

Warm-up:

$$\int \frac{\sin^2 x - \cos^2 x}{\cos x}\,dx \to \int \frac{1-\cos^2 x - \cos^2 x}{\cos x}\,dx$$

$$\to \int \frac{1-2\cos^2 x}{\cos x}\,dx \to \int (\sec x - 2\cos x)\,dx$$

$$\to \boxed{\ln|\sec x + \tan x| - 2\sin x \; dx}$$

---

1. $\lim\limits_{x\to 1} \frac{4x}{x^2+4} \to \boxed{\frac{4}{5}}$

2. $\lim\limits_{x\to 2} \frac{-x-2}{x^2-4} \to \lim\limits_{x\to 2} \frac{-(\cancel{x-2})}{(\cancel{x-2})(x+2)} \to \boxed{\frac{-1}{4}}$

3. $\lim\limits_{x\to 3} \frac{x-3}{\sqrt{x+6}-3}\left(\frac{\sqrt{x+6}+3}{\sqrt{x+6}+3}\right)$

$\to \lim\limits_{x\to 3} \frac{(\cancel{x-3})(\sqrt{x+6}+3)}{\cancel{x+6-9}} \to \cancel{x-3}$

$\to \sqrt{9}+3 \to \boxed{6}$

Expand every "ln" before deriving

Derivative of Inverse of function

Derivatives for ~~bases~~ other than e

Completing the square

Quiz Stuff (Ch 5)

## Indeterminate Forms:

1. Try direct ~~substitution~~, you end up with $\frac{0}{0}$
2. " " ", you get $\frac{\pm\infty}{\pm\infty}$

## L'Hôpital's Rule -

Use on indeterminate limits

If $\lim\limits_{x\to c} \frac{f(x)}{g(x)} = \frac{0}{0}$, then $\lim\limits_{x\to c} \frac{f(x)}{g(x)} = \lim\limits_{x\to c} \frac{f'(x)}{g'(x)}$,

repeat if you end up with $\frac{0}{0}$ again, a few times. Then Indeterminate

Test: Must write L'H when doing L'Hopital's Rule

Ex 1: $\lim\limits_{x\to 0} \frac{e^{2x}-1}{x} \to \frac{0}{0} \to$ L'Hôpital's Rule

$\to (e^{2x})(2) \to \frac{2e^{2x}}{1} \to \lim\limits_{x\to 0} 2e^{2x} \to 2e^{0} \to \boxed{2}$

Ex 2: $\lim\limits_{x\to\infty} \frac{\ln x}{x} \to \frac{\ln(\infty)}{\infty} \to \frac{\infty}{\infty}$ L'Hopital's

$\to \frac{\frac{1}{x}}{1} \to \frac{1}{x} \to \lim\limits_{x\to\infty} \frac{1}{x} \to \frac{1}{\infty} = \boxed{0}$

Ex 3:

$\lim\limits_{x\to-\infty} \frac{x^2}{e^{-x}} = \frac{\infty}{\infty}$ L'Hopital's Rule

$\to \lim\limits_{x\to-\infty} \frac{2x}{-e^{-x}} \to \frac{-\infty}{-\infty}$ L'H $\to \lim\limits_{x\to-\infty} \frac{2}{e^{-x}} = \frac{2}{\infty}$

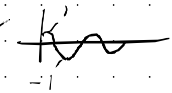
Ex 4: $\lim\limits_{x\to\infty} e^{-x}\sqrt{x} \to 0\sqrt{\infty} \to 0\cdot\infty$ L'H $\to \boxed{0}$

$\to \lim\limits_{x\to\infty} \frac{\sqrt{x}}{e^{x}} \to \lim\limits_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{e^{x}} \to \frac{0}{\infty} \to \boxed{0}$

# Chapter 5 Review

$y=\tan^{-1}x \rightarrow D:(-\infty,\infty)\quad R:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\frac{d}{dx}[\sin^{-1}u]=\frac{u'}{\sqrt{1-u^2}}$

$y=\cos^{-1}x \rightarrow D:[-1,1]\quad R:[0,\pi]$ 


$y=\sin^{-1}x \rightarrow D:[-1,1]\quad R:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$A=P(1+\frac{r}{n})^{nt}\quad A=Pe^{rt}$

$\frac{d}{dx}[\log_a u] \rightarrow \frac{u'}{u(\ln(a))}\quad \int a^x\,dx \rightarrow \frac{1}{\ln a}a^x+C$

$\frac{d}{dx}[a^u]\rightarrow(\ln a)a^u\cdot u'\quad \frac{d}{dx}[a^x]\rightarrow(\ln a)a^x$

$a^x=e^{\ln a\,x}\quad b^{\log_b u}=u\quad \int a^u\,du \rightarrow \frac{1}{\ln(a)}a^u+C$

$\frac{d}{dx}[\log_a x]=\frac{1}{\ln a\,x}\quad y=\cos^{-1}x \rightarrow D:[-1,1]\quad R:[0,\pi]$

$\int \tan(u)\,du \rightarrow -\ln|\cos u|+C\quad \int a^u\,du \rightarrow \frac{1}{\ln a}a^u+C$

$\frac{d}{dx}[\log_a x]\rightarrow\frac{1}{\ln a\,x}\quad \frac{d}{dx}[\log_a u]\rightarrow\frac{u'}{u\ln a}$

$\rightarrow$ derivative of a log with weird base is $u'/$"$u\ln a$"

$\frac{d}{dx}[\sin^{-1}u]\rightarrow\frac{u'}{\sqrt{1-u^2}}\quad y=\tan^{-1}x \rightarrow D:(-\infty,\infty)\quad R:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\int a^u\,du \rightarrow \frac{1}{\ln a}a^u+C\quad \frac{d}{dx}[\log_a x]\rightarrow\frac{1}{x\ln a}$

$\frac{d}{dx}[\log_a u]\rightarrow\frac{u'}{u(\ln(a))}\quad \frac{d}{dx}[\sin^{-1}u]\rightarrow\frac{u'}{\sqrt{1-u^2}}$

$\frac{d}{dx}[\cos^{-1}u]\rightarrow\frac{-u'}{\sqrt{1-u^2}}\quad y=\cos^{-1}x \rightarrow D:[-1,1]\quad R:[0,\pi]$

$\frac{d}{dx}[a^x]\rightarrow(\ln a)a^x\quad \int a^x\,dx \rightarrow \frac{1}{\ln a}a^x+C$

$\frac{d}{dx}[a^u] \to (\ln a) a^u u'$ ✓ $a^x = e^{\ln a x}$ ✓ $b^{\log_b u} = u$ ✓

$\int a^u du \to \frac{1}{\ln a} a^u + C$ $\frac{d}{dx}[\log_a u] \to \frac{u'}{u \ln a}$ ✗

$\frac{d}{dx}[\log_a x] \to \frac{1}{x \ln a}$ $\frac{d}{dx}[\sin^{-1} x] \to \frac{u'}{\sqrt{1-u^2}}$ ✓

$\frac{d}{dx}[a^x] \to (\ln a) a^x$ ✓ $\int \tan(u) du \to -\ln|\cos(u)| + C$ ✓

$\frac{d}{dx}[a^u] = (\ln a) a^u (u')$ ✗ $\frac{d}{dx}[\log_a u] \to \frac{u'}{u \ln a}$ ✓

$y = \tan^{-1} x \to D: (-\infty, \infty)$ $R: [-\frac{\pi}{2}, \frac{\pi}{2}]$ ✓ $\frac{d}{dx}[\cos^{-1} u] \to \frac{-u'}{\sqrt{1-u^2}}$ ✓

$\frac{d}{dx}[a^u] \to (\ln a) a^u (u')$ ✓ $\int e^x dx \to e^x + C$ ✓

$\frac{d}{dx}[\tan^{-1} u] \to \frac{u'}{1+u^2}$ ✓ $\int \frac{1}{x} dx \to \ln|x| + C$ ✓

$\int a^u du \to \frac{1}{\ln a} a^u + C$ ✓ $\frac{d}{dx}[\cot^{-1} u] \to \frac{-u'}{1+u^2}$ ✓

$e^a \cdot e^b = e^{a+b}$ $\frac{e^a}{e^b} = e^{a-b}$ ✗ $\frac{d}{dx}[\log_a x] \to \frac{1}{x \ln a}$ ✓

$\frac{d}{dx}[a^x] \to (\ln a) a^x$ ✓ $y = \cos^{-1} x \to D: [-1, 1]$ $R: [0, \pi]$ ✓

$\int a^x dx \to \frac{1}{\ln(a)} a^x + C$ ✓ $\frac{d}{dx}[\log_a u] \to \frac{u'}{u(\ln a)}$ ✓

$\frac{d}{dx}[\sec^{-1} u] \to \frac{u'}{|u|\sqrt{u^2-1}}$ ✓ $\frac{d}{dx}[a^u] \to (\ln a) a^u (u')$ ✓

$\frac{d}{dx}[\cos^{-1} u] \to \frac{-u'}{\sqrt{1-u^2}}$ ✓ $\frac{d}{dx}[\tan^{-1} u] \to \frac{u'}{1+u^2}$ ✓

$\frac{d}{dx}[\sin^{-1} u] \to \frac{u'}{\sqrt{1-u^2}}$ ✓ $\int a^u du \to \frac{1}{\ln a} a^u + C$ ✓

$\frac{d}{dx}[\cot^{-1} u] \to \frac{-u'}{1+u^2}$ ✓ $\frac{d}{dx}[a^x] \to (\ln a) a^x$ ✓

$\frac{d}{dx}[\log_a x] \to \frac{1}{x\ln a}$ ✓ $\frac{d}{dx}[\log_a u] \to \frac{u'}{u\ln a}$ ✓

$\int \tan(u)\,du \to -\ln|\cos(u)| + c$ ✓ $\frac{d}{dx}[\sec^{-1} u] \to \frac{u'}{|u|\sqrt{u^2-1}}$ ✓

$\frac{d}{dx}[a^u] \to (\ln a)a^u(u')$ ✓ $e^a \cdot e^b = e^{a+b}$ ✓ $e^a/e^b = e^{a-b}$ ✓

$\int a^x\,dx \to \frac{1}{\ln a} a^x + c$ ✓ $\int \sin(u)\,du \to -\cos(u) + c$ ✗

Integral of sin(u) is $-\cos(u) + c$ $\frac{d}{dx}[\cos^{-1}(u)] \to \frac{-u'}{\sqrt{1-u^2}}$

$\frac{d}{dx}[\tan^{-1} u] \to \frac{u'}{1+u^2}$ ✓ $\int a^u\,du \to \frac{1}{\ln a} a^u + c$ ✓

$\frac{d}{dx}[\csc^{-1} u] \to \frac{-u'}{|u|\sqrt{u^2-1}}$ ✓ $\frac{d}{dx}[\cot^{-1} u] \to \frac{-u'}{1+u^2}$ ✓

$\frac{d}{dx}[a^u] \to \ln a\, a^u u'$ ✓ $\int \csc(u)\,du \to -\ln|\csc(u) + \cot(u)| + c$ ✗

the integral of csc(u) is negative natural log of the absolute value of csc u + cot u + C

$\frac{d}{dx}[a^x] = (\ln a)a^x$ ✓ $\frac{d}{dx}[\log_a x] \to \frac{1}{x\ln a}$ ✓ $\frac{d}{dx}[\log_a u] = \frac{u'}{u(\ln a)}$ ✓

$\frac{d}{dx}[\sec^{-1} u] \to \frac{u'}{|u|\sqrt{u^2-1}}$ ✓ $\int \frac{du}{\sqrt{a^2-u^2}} \to \sin^{-1}\frac{u}{a} + c$ ✓

$\int \cos u\,du \to \sin u + c$ ✓ $\frac{d}{dx}[\csc^{-1} u] \to \frac{-u'}{|u|\sqrt{u^2-1}}$ ✓

$\frac{d}{dx}[a^u] \to (\ln a)a^u(u')$ ✓ $\int \frac{du}{a^2+u^2} \to \frac{1}{a}\tan^{-1}\frac{u}{a} + c$ ✓

$\frac{d}{dx}[\csc^{-1} u] \to \frac{-u'}{|u|\sqrt{u^2-1}}$ ✓ $\frac{d}{dx}[\tan^{-1} u] \to \frac{u'}{1+u^2}$ ✓

$\int \sin(u)\,du \to -\cos(u) + c$ ✓ $\int \frac{du}{a^2+u^2} \to \frac{1}{a}\tan^{-1}\frac{u}{a} + c$ ✓

$\int a^u\,du \to \frac{1}{\ln a} a^u + c$ ✓ $\frac{d}{dx}[\cot^{-1} u] \to \frac{-u'}{1+u^2}$ ✓

$\int \frac{du}{\sqrt{a^2-u^2}} du \to \sin^{-1}\frac{u}{a} + c$ ✓ $e^a \cdot e^b \to e^{a+b}$ ✓ $\frac{e^a}{e^b} = e^{a-b}$ ✓

$\int \csc(u)\, du \to -\ln|\csc u + \cot u| + c$ ✗

$\int \cos u\, du \to \sin u + c$ ✓ $\frac{d}{dx}[a^u] \to \ln a\, a^u u'$ ✓

$\int \frac{du}{a^2+u^2} \to \frac{1}{a}\tan^{-1}\frac{u}{a} + c$ ✓ $\frac{d}{dx}[\csc^{-1} u] \to \frac{-u'}{|u|\sqrt{u^2-1}}$ ✓

$\int a^u du \to \frac{1}{\ln a} a^u + c$ ✓ $\int \sin(u) du \to -\cos(u) + c$ ✓

$\frac{d}{dx}[\ln(x)] \to \frac{1}{x}$ ✓ $\int \frac{du}{u\sqrt{u^2-a^2}} \to \frac{1}{a}\sec^{-1}\frac{|u|}{a} + c$ ✓

$\int \frac{1}{u} du \to \ln|u| + c$ ✗ 1. $\cos^{-1}\frac{u}{a} + c$ ✓ 2. $\frac{1}{a}\cot^{-1}\frac{u}{a} + c$ ✓

3. $\frac{1}{a}\csc^{-1}\frac{|u|}{a} + c$ ✓ $y = e^x$ iff $x = \ln y$ ✓

$x^2 - 4x + 7 \to (-4/2)^2 \to -2^2 \to 4 \to (x^2 - 4x + 4) + 7 - 4$

$\to (x-2)(x-2) + 3 \to (x-2)^2 + 3$ ✓

$\int \frac{du}{a^2+u^2} \to \frac{1}{a}\tan^{-1}\frac{u}{a} + c$ ✓ $\int \frac{du}{u\sqrt{u^2-a^2}} \to \frac{1}{a}\sec^{-1}\frac{|u|}{a} + c$ ✓

$\int \frac{du}{\sqrt{a^2-u^2}} \to \sin^{-1}\frac{u}{a} + c$ $\int \csc(u)\, du \to -\ln|\csc(u) + \cot(u)| + c$ ✓

$\int a^u du \to \frac{1}{\ln a} a^u + c$ ✓ 1. $\cos^{-1}\frac{u}{a} + c$ ✓ 2. $\frac{1}{a}\cot^{-1}\frac{u}{a} + c$

3. $\frac{1}{a}\csc^{-1}\frac{|u|}{a} + c$ ✗ $\frac{d}{dx}[\csc^{-1} u] \to \frac{u'}{|u|\sqrt{u^2-1}}$ ✗ $\int \frac{1}{u} du \to \ln|u| + c$ ✓

$\to (x-2)^2 + 3$ ✓ $\int \cos(u) du \to \sin(u) + c$ ✓ $g'(x) = \frac{1}{f'(g(x))}$

$\frac{d}{dx}[\csc^{-1} u] \to \frac{-u'}{|u|\sqrt{u^2-1}}$ ✓ $\int \sin(u) du \to -\cos(u) + c$ ✓

1. $\cos^{-1}\frac{u}{a} + c$ ✓ 2. $\frac{1}{a}\cot\frac{u}{a} + c$ ✓ 3. $\frac{1}{a}\csc^{-1}\frac{|u|}{a} + c$ ✓

$\int \frac{du}{a^2+u^2} \to \frac{1}{a}\tan^{-1}\frac{u}{a}+c$ ✓ $\frac{d}{dx}[\csc^{-1}u] \to \frac{-u'}{|u|\sqrt{u^2-1}}$ ✓

$\int \frac{du}{u\sqrt{u^2-a^2}} \to \frac{1}{a}\sec^{-1}\frac{|u|}{a}+c$ ✓ $\int \csc(u)\,du \Rightarrow$

$\int \csc(u)\,du \to -\ln|\csc(u)+\cot(u)|+c$ ✓ $\to x^2-4x+7 \to (x^2-4x+\_\_)+7-\_\_$

$\to (-4/2)^2 \to -2^2 \to 4 \to (x^2-4x+4)+3 \to (x-2)(x-2)+3 \to (x-2)^2+3$ ✓

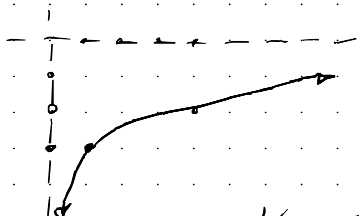
1. $\cos\frac{u}{a}+c$ ✓ 2. $\frac{1}{a}\cot^{-1}\frac{u}{a}+c$ ✓ 3. $\frac{1}{a}\csc^{-1}\frac{|u|}{a}+c$ ✓

$\frac{d}{dx}[\csc^{-1}u] \to \frac{-u'}{|u|\sqrt{u^2-1}}$ ✓ $\int \frac{1}{u}\,du \to \ln|u|+c$ ✓

$\int \cos(u)\,du \to \sin(u)+c$ ✓

## Chapter 5 Review Exercises

(1) $f(x)=\ln(x)-3$



Shifted down 3 units

ln graph

(3) $\ln\sqrt[5]{\frac{4x^2-1}{4x^2+1}} \to \ln\left(\frac{4x^2-1}{4x^2+1}\right)^{1/5} \to \frac{1}{5}\left(\ln\frac{(2x-1)(2x+1)}{4x^2+1}\right)$

$\to \frac{1}{5}\left(\left(\ln(2x-1)+\ln(2x+1)\right)-\ln(4x^2+1)\right)$

(5) $\ln 3+\frac{1}{3}\ln(4-x^2)-\ln x \to \ln 3+\ln\sqrt[3]{4-x^2}-\ln x$

$\to \ln\frac{3(\sqrt[3]{4-x^2})}{x}$

(7) $g(x)=\ln\sqrt{2x} \to g'(x)=\frac{1}{2}\ln 2x \to \frac{1}{2}\left(\frac{2}{2x}\right) \to \frac{1}{2x}$

(9) $f(x)=x\sqrt{\ln x} \to f'(x)=\sqrt{\ln x}+x\left(\frac{1}{2}(\ln x)^{-1/2}\left(\frac{1}{x}\right)\right)$

$\to \sqrt{\ln x}+\frac{x}{2}(\ln x)^{-1/2}\left(\frac{1}{x}\right) \to \sqrt{\ln x}+\frac{1}{2\sqrt{\ln x}}$

$\to \frac{1}{2\sqrt{\ln x}}+\left(\frac{\sqrt{\ln x}}{1}\cdot\frac{2\sqrt{\ln x}}{2\sqrt{\ln x}}\right) \to \frac{1}{2\sqrt{\ln x}}+\frac{2(\ln x)}{2\sqrt{\ln x}}$

$\to \frac{1+2(\ln x)}{2\sqrt{\ln x}}$

(11) $y = \ln\sqrt{\frac{x^2+4}{x^2-4}} \to \frac{1}{2}\left[(\ln x^2+4) - (\ln x^2-4)\right]$

$\to \frac{1}{2}\left[\left(\frac{2x}{x^2+4}\right) - \left(\frac{2x}{x^2-4}\right)\right] \to \frac{1}{2}\left[\frac{2x(x^2-4)}{x^4-16} - \frac{2x(x^2+4)}{x^4-16}\right]$

$\to \frac{1}{2}\left[\frac{2x^3-8x}{x^4-16} - \frac{2x^3+8x}{x^4-16}\right] \to \frac{1}{2}\left[\frac{-16x}{x^4-16}\right] \to \boxed{\frac{-8x}{x^4-16}}$

(13) $y = \ln(2+x) + \frac{2}{2+x}$ @ $(-1,2)$ $\to 2u^{-1}$

$u = 2+x \quad u' = 1$

$\to \frac{1}{2+x} + (-2u^{-2}) \to \frac{1}{2+x} - \frac{2}{(2+x)^2} \to \frac{2+x}{(2+x)^2} - \frac{2}{(2+x)^2}$

$\to \frac{x}{(2+x)^2} \to y'\Big|_{(-1,2)} \to \frac{-1}{(2+(-1))^2} \to \frac{-1}{1^2} \to -1$

$\to y-2 = -1(x+1) \to \boxed{y = -x+1}$

(15) $\int \frac{1}{7x-2}\,dx \to u = 7x-2 \quad du = 7\,dx$

$\to \frac{1}{7}\int \frac{1}{u}\,du \to \frac{1}{7}\ln|u| + C \to \boxed{\frac{1}{7}\ln|7x-2| + C}$

(17) $\int \frac{\sin x}{1+\cos x}\,dx \quad u = \cos x + 1 \quad du = -\sin dx$

$\to -\int \frac{1}{u}\,du \to -\ln|u| + C \to \boxed{-\ln|\cos x + 1| + C}$

(19) $\int_1^4 \frac{2x+1}{2x}\,dx \to \int_1^4 \frac{2x}{2x}\,dx + \int_1^4 \frac{1}{2x}\,dx = \frac{1}{2}\int \frac{1}{x}\,dx$

$\to \int_1^4 1\,dx + \int_1^4 \frac{1}{2x}\,dx \to (x) + \left(\frac{1}{2}\int \frac{1}{x}\,dx\right)$ $\quad \frac{1}{2}(0) \to 0$

$\to \left[x + \frac{1}{2}\ln|x|\right]_1^4 \to \left(4 + \frac{1}{2}\ln 4\right) - \left(1 + \cancel{\frac{1}{2}\ln 1}\right)$

$\to 4 + \frac{1}{2}\ln 4 - 1 \to 3 + \ln\sqrt{4} \to \boxed{3 + \ln 2}$

(21) $\int_0^{\pi/3} \sec\theta\,d\theta \to \left[\ln|\sec\theta + \tan\theta|\right]_0^{\pi/3}$

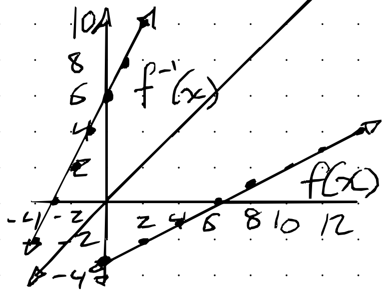
$\to \left(\ln\left|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right|\right) - \left(\ln|\sec 0 + \tan 0|\right)$

$\to \ln|2 + \sqrt{3}| - \cancel{(\ln|1+0|)} \to \ln|2+\sqrt{3}|$

(23) $f(x) = \frac{1}{2}x - 3$

(a) $x = \frac{1}{2}y - 3 \Rightarrow x + 3 = \frac{1}{2}y \Rightarrow \boxed{2x + 6 = y}$ $f^{-1}(x)$

(b)




(c) $\frac{1}{2}(2x + 6) - 3 \Rightarrow x + 3 - 3 \Rightarrow x$

$2(\frac{1}{2}x - 3) + 6 \Rightarrow x - 6 + 6 \Rightarrow x$

(d) $f$ $D: (-\infty, \infty)$ $R: (-\infty, \infty)$ $f^{-1}$ $D: (-\infty, \infty)$ $R: (-\infty, \infty)$

(27) $f(x) = \sqrt[3]{x+1}$

$\Rightarrow x = \sqrt[3]{y+1} \Rightarrow x^3 = y + 1 \Rightarrow y = x^3 - 1$

(29) $f(x) = x^3 + 2$, $a = -1$

$f(?) = -1$ $f^{-1}(-1) = ?$

$-1 = x^3 + 2 \Rightarrow -3 = x^3 \Rightarrow \sqrt[3]{-3} = x$

$\Rightarrow f((-3)^{1/3}) = -1$ $f^{-1}(-1) = (-3)^{1/3}$

$\Rightarrow 3x^2$ $\frac{1}{3(-3^{1/3})^2} \Rightarrow \frac{1}{3(-3^{2/3})}$ $f'(-3^{1/3})$

$\Rightarrow 3(-3^{1/3})^2 \Rightarrow 3(-3^{2/3})$

Formula Check

$\int \sin(u)\,du = -\cos u + c$ ✓ $\frac{d}{dx}[\sin^{-1} u] \Rightarrow \frac{u'}{\sqrt{1-u^2}}$ ✓

$\frac{d}{dx}[\csc^{-1} u] \Rightarrow \frac{-u'}{|u|\sqrt{u^2-1}}$ ✓ $g'(x) = \frac{1}{f'(g(x))}$ ✓

$A = P\cdot(1+\frac{r}{n})^{nt}$ $A = Pe^{rt}$ ✓ $\frac{d}{dx}[\cot^{-1} u] \to \frac{-u'}{1+u^2}$ ✓

$\frac{d}{dx}[\cos^{-1} u] \to \frac{-u'}{\sqrt{1-u^2}}$ ✓ $e^a \cdot e^b = e^{a+b}$ $\frac{e^a}{e^b} = e^{a-b}$ ✓

$\frac{d}{dx}[\log_a u] = \frac{u'}{u(\ln a)}$ ✓ $\frac{d}{dx}[\sec^{-1} u] \to \frac{u'}{|u|\sqrt{u^2-1}}$ ✓

$\int e^u du \to e^u \cdot u' + c$ ✓ $\int \sec(u)\, du = \ln|\sec u + \tan u| + c$ ✓

$y = \cos^{-1} u$ $D: [-1, 1]$ $R: [0, \pi]$ ✓ $y = \sin^{-1} x$ $D: [-1, 1]$ $R: [-\frac{\pi}{2}, \frac{\pi}{2}]$ ✓

$\frac{d}{dx}[\tan^{-1} u] \to \frac{u'}{1+u^2}$ ✓ $1: \cos^{-1} \frac{u}{a} + c$ ✓ $2. \frac{1}{a} \cot^{-1} \frac{u}{a} + c$ ✓

$3. \frac{1}{a} \csc^{-1} \frac{|u|}{a} + c$ ✓ $\int \tan(u)\, du \to -\ln|\cos(u)| + c$ ✓

$x^2 - 4x + 7 \to (x^2 - 4x + \_\_) + 7 - \_\_ \to (-4/2)^2 \to -2^2 \to 4$

$\to (x^2 - 4x + 4) + 7 - 4 \to (x-2)^2 + 3$ ✓ $\int \frac{1}{x} dx \to \ln|x| + c$ ✓

$\frac{d}{dx}[\log_a x] \to \frac{1}{x \ln a}$ ✓ $\int \frac{du}{u\sqrt{u^2 - a^2}} \to \frac{1}{a} \sec^{-1} \frac{|u|}{a} + c$ ✓

$\frac{d}{dx}[a^u] \to (\ln a) a^u (u')$ ✓ $\int e^x dx \to e^x + c$ ✓

$\frac{d}{dx}[\sin^{-1} u] \to \frac{u'}{\sqrt{1-u^2}}$ ✓ $\log_b a \to \frac{\ln a}{\ln b}$

One to one (passes both horizontal & vertical line tests) ✓

Monotonic (strictly increasing <u>or</u> decreasing) ✓

$\int \frac{1}{u} du \to \ln|u| + c$ ✓ $\int \cos(u)\, du \to \sin u + c$ ✓

$\frac{d}{dx}[\csc^{-1} u] \to \frac{-u'}{|u|\sqrt{u^2-1}}$ ✓ $\frac{d}{dx}[\cos^{-1} u] \to \frac{-u'}{\sqrt{1-u^2}}$ ✓

$e^a \cdot e^b = e^{a+b}$ ✓ $\frac{e^a}{e^b} = e^{a-b}$ ✓ $\frac{d}{dx}[\cot^{-1} u] \to \frac{-u'}{1+u^2}$ ✓

$\frac{1}{f'(g(x))}$ ✓ $A=P(1+\frac{r}{n})^{nt}$ ✓ $A=Pe^{rt}$ ✓

$\frac{d}{dx}[\log_a u] \to \frac{u'}{u(\ln(a))}$ ✓ $y=\cos^{-1}x$ $D:[-1,1]$ $R:[0,\pi]$ ✓

$\int \csc(u)\,du \to -\ln|\csc u+\cot u|+c$ ✓ $\frac{d}{dx}[\ln u] \to \frac{u'}{u}$ ✓

$\int \sec(u)\,du \to \ln|\sec(u)+\tan(u)|+c$ ✓ $y=\sin^{-1}x$ ✓ $D:[-1,1]$ $R:[-\frac{\pi}{2},\frac{\pi}{2}]$

$\frac{d}{dx}[\tan^{-1}u] \to \frac{u'}{1+u^2}$ ✓ $\frac{d}{dx}[e^x] \to e^x$ ✓ $\frac{d}{dx}[e^u] \to e^u \cdot u'$ ✓

$a^x = e^{(\ln a)x}$ ✓ $b^{\log_b u}=u$ ✓ $\int \tan(u)\,du \to -\ln|\cos(u)|+c$ ✓

1. $\cos^{-1}\frac{u}{a}+c$ ✓ 2. $\frac{1}{a}\cot^{-1}\frac{u}{a}+c$ ✓ 3. $\frac{1}{a}\csc^{-1}\frac{|u|}{a}+c$ ✓

$\int \frac{du}{\sqrt{a^2-u^2}} \to \sin^{-1}\frac{u}{a}+c$ ✓ $\int \frac{du}{u\sqrt{u^2-a^2}} \to \frac{1}{a}\sec^{-1}\frac{|u|}{a}+c$ ✓

$\frac{d}{dx}[\ln x] \to \frac{1}{x}$ ✓ $\int a^u\,du \to \frac{1}{(\ln(a))}a^u+c$ ✓

$\frac{d}{dx}[a^x] \to (\ln a)a^x$ ✓ $\int a^x\,dx \to \frac{1}{\ln a}a^x+c$ ✓

$\int \cot u\,du \to \ln|\sin(u)|+c$ ✓ $y=\tan^{-1}x$ $D:(-\infty,\infty)$ ✓ $R:[-\frac{\pi}{2},\frac{\pi}{2}]$ ✓

$\frac{d}{dx}(y=x^{x-1}) \to \ln y = x-1\ln x$

$\to \frac{y'}{y} = ((x-1)(\frac{1}{x}))+((1)(\ln x))$

$\to y' = y(x-1(\frac{1}{x})+\ln x) \to y' = x^{x-1}(x-1(\frac{1}{x})+\ln x)$ ✓

$y=e^x$ iff $x=\ln y$ ✓ $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}+c$ ✓

$\int \sin(u)\,du \to -\cos u+c$ ✓ $\frac{d}{dx}[\sin^{-1}u] \to \frac{u'}{\sqrt{1-u^2}}$ ✓

$\int \csc(u)\,du = -\ln|\csc u+\cot u|+c$ ✓ $\int \sec(u)\,du = \ln|\sec u+\tan u|+c$ ✓

$\frac{d}{dx}[\ln(u)] \to \frac{u'}{u}$ ✓ $\frac{d}{dx}[\cos^{-1}u] \to \frac{-u'}{\sqrt{1-u^2}}$ ✓

$\int \tan(u)\,du \to -\ln|\cos(u)| + c$ ✓

$\frac{d}{dx}[\csc^{-1} u] \to \frac{-u'}{|u|\sqrt{u^2-1}}$ ✓

$\int \cot(u)\,du \to \ln|\sin(u)| + c$ ✓

$\int \csc(u)\,du \to -\ln|\csc u + \cot u| + c$ ✓

$\int \sin(u)\,du \to -\cos u + c$ ✓

$\int \sec(u)\,du \to \ln|\sec(u) + \tan(u)| + c$ ✓

# Chapter 5 Quiz Review

**ln differentiation:** $y = x^{(x-1)}$

→ you have a function of $x$ to the power of a function of $x$: $y = g(x)^{f(x)}$

1. take ln of both sides
2. differentiate both sides
3. isolate $y'$ → 4. replace $y$ with given

(10) $\int \frac{x^2 - 3x + 9}{x - 3}\,dx$ If numerator's degree is ≥ degree of denom, use division

$$\to \begin{array}{c|ccc} 3 & 1 & -3 & 9 \\ & & 3 & 0 \\ \hline & 1 & 0 & 9 \end{array} \to \int x\,dx + 9\int \frac{1}{x-3}\,dx$$ synth. division

$$\to \boxed{\frac{x^2}{2} + 9\ln|x-3| + C}$$

(12) $\int 8^{3x}\,dx$ $u = 3x$ $du = 3\,dx$

$$\to \frac{1}{3}\int a^u\,du \to \frac{1}{3}\left(\frac{1}{\ln 8}\right)8^{3x} + C \to \frac{1}{3(\ln 8)}8^{3x} + C$$

(8) $\int x \csc(x^2)\, dx \quad u = x^2 \quad du = 2x\,dx$

$\frac{1}{2}\int \csc(u)\, du \rightarrow -\frac{1}{2}\ln|\csc u + \cot u| + c$

$\rightarrow -\frac{1}{2}\ln|\csc x^2 + \cot x^2| + C$ replace the "u" from u substitution

(9) $\int \frac{-4}{x^2+6x+13}\,dx \rightarrow (x^2+6x+9)+13-9$

$\rightarrow (x+3)(x+3)+4$

$\rightarrow \int \frac{-4}{(x+3)^2+4}\,dx$

$\frac{1}{a}\tan^{-1}\frac{u}{a}+C$

$\rightarrow u = x+3 \quad a = 2 \quad u' = 1 \quad du = 1\,dx$

$-4\left(\frac{1}{2}\tan^{-1}\frac{x+3}{2}\right) + C \rightarrow -2\tan^{-1}\left(\frac{x+3}{2}\right) + C$

(16) $\int -3\sec x\, dx \quad u = x \quad du = 1\,dx$

$\rightarrow -3\int \sec x\, dx \quad -3\ln|\sec x + \tan x| + C$

AP

$f(x) = 2xe^{2x}$

(a) find $\lim_{x\to\infty} f(x)$ & $\lim_{x\to-\infty} f(x)$

$\rightarrow \lim_{x\to-\infty} = 0 \qquad \lim_{x\to\infty} = \infty$

(b) $f'(x) = 2e^{2x} + 2x(2)(e^{2x}) = 2e^{2x}(1+2x)$

$f'(-\frac{1}{2}) = 0 \quad f(-\frac{1}{2}) = -\frac{1}{e}$

$f'(x) < 0$ on $x < -\frac{1}{2}$, $f'(x) > 0$ on $x > -\frac{1}{2}$

(c) $[-\frac{1}{e}, \infty)$

(d) $y' = be^{bx} + b^2xe^{bx} = be^{bx}(1+bx)$

$y'(-\frac{1}{b}) = 0$ @ $x = -\frac{1}{b}$ $y = -\frac{1}{e}$

$y = bxe^{bx}$ y has abs. min. of $-\frac{1}{e}$ for all nonzero "b"

Formula review

$\frac{d}{dx}(\ln(x)) \Rightarrow \frac{1}{x}$ $\frac{d}{dx}(\ln(u)) \Rightarrow \frac{u'}{u}$

$\int \frac{1}{x}\,dx \Rightarrow \ln|x| + c$ ✓ $\int \frac{1}{u}\,du \Rightarrow \ln|u| + c$ ✓

$\frac{d}{dx}(y = x^{x-1}) \Rightarrow \ln y = (x-1)\ln x$

$\Rightarrow \frac{y'}{y} = (x-1)(\frac{1}{x}) + (1)(\ln x)$

$\Rightarrow \frac{y'}{y} = \frac{x-1}{x} + \ln x \Rightarrow y' = y(\frac{x-1}{x} + \ln x)$

$y' = (x^{x-1})(\frac{x-1}{x} + \ln x)$

$\int \sin(u)\,du \Rightarrow -\cos u + c$ ✓ $\int \cos u\,du \Rightarrow \sin u + c$ ✓

$\int \tan u\,du = -\ln|\cos u| + c$ ✓ $\int \cot u\,du \Rightarrow \ln|\sin u| + c$ ✓

$\int \sec u\,du \Rightarrow \ln|\sec u + \tan u| + c$ ✓ One-to-one ✓

Monotonic ✓

$\int \csc u\,du \Rightarrow -\ln|\csc u + \cot u| + c$ ✓

$g'(x) = \frac{1}{f'(g(x))}$ ✓ $y = e^x$ iff $x = \ln y$ ✓

$e^a \cdot e^b = e^{a+b}$ ✓ $\frac{e^a}{e^b} = e^{a-b}$ ✓ $\frac{d}{dx} e^x \Rightarrow e^x$ ✓

$\frac{d}{dx} e^u = e^u u'$ ✓ $\int e^x\,dx \Rightarrow e^x + C$ $\int e^u\,du \Rightarrow e^u \cdot u' + C$

$\int \frac{1}{x}\,dx \Rightarrow \ln|x| + c$ ✓ $\frac{d}{dx}(\ln(u)) \Rightarrow \frac{u'}{u}$ ✓

$a^x = e^{\ln a x}$ $\quad b^{\log_b u} = u$

$\int \frac{1}{u}\,du \Rightarrow \ln|u| + C \quad \int e^x\,dx \Rightarrow e^x + C$

$\int e^u\,du \Rightarrow e^u u' + C \quad \int \cos u\,du \Rightarrow \sin u + C$

$\int \tan u\,du \Rightarrow -\ln|\cos u| + C \quad \int \sin(u)\,du \Rightarrow -\cos u + C$

$\int \cot(u)\,du \Rightarrow \ln|\sin(u)| + C \quad \int \csc u\,du \Rightarrow -\ln|\csc u + \cot u| + C$

$\int \sec u\,du \Rightarrow \ln|\sec(u) + \tan(u)| + C \quad g'(x) = \frac{1}{f'(g(x))}$

monotonic one-to-one $\quad y = e^x$ ~~iff~~ $x = \ln|y|$

$\frac{d}{dx} e^x \Rightarrow e^x \quad e^a \cdot e^b = e^{a+b} \quad \frac{e^a}{e^b} = e^{a-b}$

$\frac{d}{dx} e^u \Rightarrow e^u u' \quad \frac{d}{dx}(\ln x) \Rightarrow \frac{1}{x} \quad \int \frac{1}{u}\,du \Rightarrow \ln|u| + C$

$\frac{d}{dx}(y = x^{x-1}) \Rightarrow \quad y' = x^{x-1}\left(\frac{x-1}{x} + \ln x\right)$

$\int \frac{1}{x}\,dx \Rightarrow \ln|x| + C \quad \frac{d}{dx}(\ln(u)) \Rightarrow \frac{u'}{u}$

$a^x = e^{\ln a x} \quad b^{\log_b u} = u \quad \int e^x\,dx \Rightarrow e^x + C$

$\int e^u\,du \Rightarrow e^u u' + C \quad \int \cos(u)\,du \Rightarrow \sin(u) + C$

$\int \sin(u)\,du \Rightarrow -\cos u + C \quad \int \tan(u)\,du \Rightarrow -\ln|\cos(u)| + C$

$\int \csc(u)\,du \Rightarrow -\ln|\csc u + \cot| + C \quad \int \cot u\,du \Rightarrow \ln|\sin(u)| + C$

$\log_b a \Rightarrow \frac{\log_{\text{desired base}} a}{\log_{\text{desired base}} b} \quad \frac{d}{dx}[a^x] \Rightarrow (\ln a)a^x$

$\frac{d}{dx}[a^u] \Rightarrow (\ln a)a^u(u') \quad \frac{d}{dx}[\log_a x] \Rightarrow \frac{1}{\ln a\, x}$

$\frac{d}{dx}[\log_a u] \Rightarrow \frac{u'}{u(\ln a)} \quad \int a^x\,dx \Rightarrow \frac{1}{\ln a} a^x + C$

$\frac{d}{dx}[a^x] \to (\ln a)a^x$ $\quad \frac{d}{dx}[a^u] \to (\ln a)a^u(u')$

$\int a^u\,du \to \frac{1}{\ln a}a^u + C$ $\quad A = Pe^{rt}$

$A = P(1+\frac{r}{n})^{nt}$

$y = \sin^{-1}(x) \to D: [-1, 1]$, $R: [-\frac{\pi}{2}, \frac{\pi}{2}]$

$y = \cos^{-1}x$ $\quad D: [-1, 1]$, $R: [0, \pi]$

$y = \tan^{-1}x$ $\quad D: (-\infty, \infty)$, $R: [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\frac{d}{dx}[\sin^{-1}(u)] \to \frac{u'}{\sqrt{1-u^2}}$ $\quad \frac{d}{dx}[\cos^{-1}u] \to \frac{-u'}{\sqrt{1-u^2}}$

$\frac{d}{dx}[\tan^{-1}x]$ $\frac{u'}{1+u^2}$ $\quad \frac{d}{dx}[\cot^{-1}u] \to \frac{-u'}{1+u^2}$

$\log_b a \to \frac{\log_{db} a}{\log_{db} b}$ $\quad \frac{d}{dx}[\sec^{-1}u] \to \frac{u'}{|u|\sqrt{u^2-1}}$

$\frac{d}{dx}[\log_a x] \to \frac{1}{\ln a\, x}$ $\quad \frac{d}{dx}[\log_a u] \to \frac{u'}{u \ln a}$

$\boxed{\frac{d}{dx}[a^u] \to (\ln a)a^u(u')}$ $\quad \int a^x\,dx \to \frac{1}{\ln a}a^x + C$

$\frac{d}{dx}[a^x] \to \ln a\; a^x$ $\quad \frac{d}{dx}[\csc^{-1}x] \to \frac{-u'}{|u|\sqrt{u^2-1}}$

$A = Pe^{rt}$ $\quad A = P(1+\frac{r}{n})^{nt}$ $\quad \int a^u\,du \to \frac{1}{\ln a}a^u + C$

$\frac{d}{dx}[a^u] \to (\ln a)a^u(u')$ $\quad \int \frac{du}{\sqrt{a^2-u^2}} \to \sin^{-1}\frac{u}{a} + C$

$\int \frac{du}{a^2+u^2} \to \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

$\int \frac{du}{u\sqrt{u^2-a^2}} \to \frac{1}{a}\sec^{-1}\frac{|u|}{a} + C$

1. $\cos^{-1}\frac{u}{a} + C$

$\frac{d}{dx}[\sin^{-1}u] \to \frac{u'}{\sqrt{1-u^2}}$

$-4/2 \to (-2)^2 \to 4$

2. $\frac{1}{a}\cot^{-1}\frac{u}{a} + C$

$x^2 - 4x + 7$

3. $\frac{1}{a}\csc^{-1}\frac{|u|}{a} + C$

$\to (x^2 - 4x + \underline{4}) + 7 - 4 \to (x-2)^2 + 3$

$\frac{d}{dx}\ln x \Rightarrow \frac{1}{x}$ $\frac{d}{dx}\ln u \Rightarrow \frac{u'}{u}$ ✓

$\int \frac{1}{x}\,dx = \ln(x) + C$ $\int \frac{1}{u}\,du \Rightarrow \ln|u| + C$

$\Rightarrow y' = x^{x-1}\left(\frac{x-1}{x} + \ln x\right)$ $\int \sin(u)\,dx \Rightarrow -\cos(u) + C$

$\int \cos u\,du \Rightarrow \sin u + C$ $\int \tan(u)\,du \Rightarrow -\ln|\cos(u)| + C$

$\int \cot u\,du \Rightarrow \ln|\sin u| + C$ $\int \sec u\,du \Rightarrow \ln|\sec u + \tan u| + C$

$\int \csc(u)\,du \Rightarrow -\ln|\csc u + \cot u| + C$ monotonic, one-to-one

$g'(x) = \frac{1}{f'(g(x))}$ $y = e^x$ <u>iff</u> $x = \ln(y)$

$e^a \cdot e^b = e^{a+b}$ $e^a / e^b = e^{a-b}$ $\frac{d}{dx}(e^x) \Rightarrow e^x$

$\frac{d}{dx} e^u \Rightarrow e^u u'$ $e^x + C$ $e^u u' + C$

$a^x = e^{\ln a\, x}$ $b^{\log_b u} = u$ $\frac{\log_d a}{\log_d b}$

$\ln a\; a^x$ $(\ln a) a^u (u')$

$\frac{1}{\ln a}$ $\frac{u'}{u \ln a}$ $\int a^x\,dx \Rightarrow \frac{1}{\ln a} a^x + C$

$\frac{1}{\ln a} a^u + C$ $A = Pe^{rt}$ $A = P\left(1 + \frac{r}{n}\right)^{nt}$

$\frac{-u'}{\sqrt{1-u^2}}$ $\frac{-u'}{1+u^2}$ $\frac{-u'}{|u|\sqrt{u^2-1}}$

$\sin^{-1}\frac{u}{a} + C$ $\frac{1}{a}\tan^{-1}\frac{u}{a} + C$ $\frac{1}{a}\sec^{-1}\frac{|u|}{a} + C$

$\frac{d}{dx}[\log_a u] \Rightarrow \frac{1}{\ln a\, x}$

# 6.1 Slope Fields

General Solution → solution with a $C$

Specific/Particular Solution → uses given point to find $C$

$\textcircled{1}\ \lim_{x\to 0} \frac{e^x-1}{4x^3} = \frac{e^0-1}{4\cdot 0^3} = \frac{0}{0}$

L'H $\lim_{x\to 0} \frac{e^x}{12x^2} \to \frac{1}{0} \to$ DNE / $\infty$

$\textcircled{2}\ \lim_{x\to 1} \frac{(x-1)}{\ln x} \to \frac{1-1}{\ln 1} = \frac{0}{0}$

L'H $\lim_{x\to 1} \frac{1}{1/x} = \frac{x}{1} = \frac{1}{1} \to 1$

$\textcircled{3}\ \lim_{x\to 0} \frac{2(e^x-1-x)}{x^2} \to \frac{0}{0}$

L'H $\lim_{x\to 0} \frac{2e^x-2-2x}{x^2} = \frac{2e^x-2}{2x} = \frac{2e^x}{2} \to \frac{2}{2} \to 1$

$\textcircled{4}\ \lim_{x\to 0} \frac{2(e^x-x)}{x^2} \overset{L'H}{=} \frac{2e^x-2x}{x^2} \to \frac{2}{0}$ — DNE/$\infty$

$\to \frac{0}{0}$

Ex 1: $y''-y=0$ } Prove whether (a, b, c) are solutions of this differential solution

ⓐ $y=\sin x \to -\sin x - \sin x \neq 0$

$\to y' = \cos x$ $\quad y=\sin(x)$ is not a solution

$- y'' = -\sin x$

$y''-y=0$

ⓑ $y = 4e^{-x}$ $\quad y''-y=0$

$y' = -4e^{-x}$ $\quad 4e^{-x} - 4e^{-x} = 0$

$y'' = 4e^{-x}$ $\quad y=4e^{-x}$ is a solution

ⓒ $y = Ce^x$

$y' = Ce^x$ $\quad y''-y=0 \to Ce^x - Ce^x = 0$

$y'' = Ce^x$

$y=Ce^x$ is a solution

Ex 2:

$xy'-3y=0$ verify $Cx^3$ is a solution. Then find particular solution by initial conditions $y=2$ & $x=-3$

$y' = 3Cx^2$ $\quad xy'-3y=0$

$x(3Cx^2) - 3(Cx^3) \to 3Cx^3 - 3Cx^3 = 0$ ✓

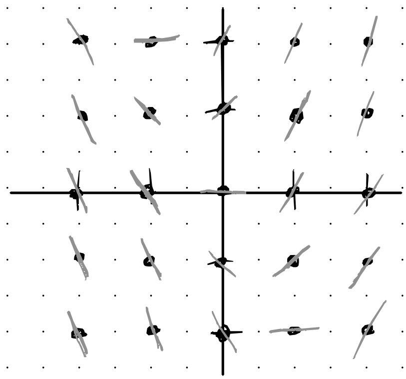
$y=Cx^3$ $\quad 2 = C(-3)^3$ $\quad 2 = C(-27)$

$\frac{2}{-27} = C$ $\quad$ partic. soln.: $y = \frac{-2}{27}x^3$

Ex 3: Sketch slope field for the differential equation

$y' = 2x + y$ use the slope field to graph solution @ $(1,1)$

| $x$ | $y$ | $y'$ |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 3 |
| 2 | 2 | 6 |
| 1 | -2 | 0 |
| 0 | 1 | 1 |
| 0 | 2 | 2 |
| 0 | -1 | -1 $\quad -1/1$ |
| 0 | -2 | -2 $\quad -2/1$ |
| 1 | -1 | 1 |
| 1 | 0 | 2 |
| 1 | 2 | 4 |



# 6.2 Growth & Decay

**Ex 1:** $y' = \frac{2x}{y}$

$$\frac{dy}{dx} = \frac{2x}{y}$$

$$dy = \frac{2x}{y}\,dx$$

$$y\,dy = 2x\,dx$$

$$\int y\,dy = \int 2x\,dx$$

$$\rightarrow \frac{y^2}{2} = x^{2+C} \rightarrow y^2 = 2x^{2+C} \rightarrow y = \pm\sqrt{2x^2 + C}$$

Warm up:

1. $\lim_{x\to 0} \frac{1-\cos(3x)}{\cos(5x)-1} \rightarrow \frac{1-\cos(0)}{\cos(0)-1} \rightarrow \frac{0}{0}$

$\underset{x\to 0}{\lim}$ L'H $\frac{3\sin(3x)}{-5\sin 5x} \rightarrow \frac{3(0)}{5(0)} \rightarrow \frac{0}{0}$

$\underset{x\to 0}{\lim}$ L'H $\frac{9\cos 3x}{-25\cos 5x} \rightarrow \frac{9(1)}{-25(1)} \rightarrow \boxed{-\frac{9}{25}}$

2. $\int \frac{e^x}{e^x+1}\,dx \rightarrow y = \ln(e^x+1) + \ln 3$

$\frac{dy}{dx} = \frac{e^x}{e^x+1}$ $\quad u = e^x + 1$, $du = e^x\,dx$ $\quad y(0) = \ln 6$

$\int dy = \int \frac{e^x}{e^x+1}\,dx$

$y = \int \frac{1}{u}\,du \rightarrow y = \ln|u| + C$

$\rightarrow y = \ln|e^x+1| + C$

$\ln 6 = \ln|e^0+1| + C \rightarrow \ln 6 = \ln 2 + C$

$\boxed{\ln\frac{6}{2} = C}$

## Exponential Growth & Decay Model

If $y$ is diff'able of $t$ such that $y > 0$ & $y' = ky$ for some constant $k$, then:

$$y = Ce^{kt}$$

$C$ is initial value of $y$, "$k$" is the proportionality constant

Rate of Change of $y$ $\rightarrow \frac{dy}{dt} \overset{\text{is}}{=} ky$ proportional to $y$

$$dt\left(\frac{dy}{dt}\right) = (ky)\,dy \rightarrow \frac{dy}{y} = \frac{(k\cancel{y})\,dt}{\cancel{y}}$$

$$\rightarrow \frac{dy}{y} = k\,dt$$

$$\rightarrow \int \frac{1}{y}\,dy = \int k\,dt \rightarrow \ln|y| = kt + C$$

$$\rightarrow e^{\ln|y|} = e^{(kt+C)}$$

$$\rightarrow y = e^{kt} \cdot \underbrace{e^{c}}_{=C} \rightarrow y = Ce^{kt}$$

If "Rate of change of y is proportional to y" is in the question, follow last process (seperation of variables)

Ex 2: $\frac{dy}{dt} = ky \Rightarrow \frac{1}{y}\,dy = k\,dt$

$\rightarrow \int \frac{1}{y}\,dy = \int k\,dt \Rightarrow \ln|y| = kt + c$

$\Rightarrow e^{\ln|y|} = e^{(kt+c)} \Rightarrow y = e^{kt} \cdot e^{c}$

$\Rightarrow y = Ce^{kt}$

Find C & k

when $t=0$, $y=2$ $\quad 2 = Ce^{k(0)} \Rightarrow 2 = C$

$C=2$ $\quad$ find k $\quad 4 = 2e^{k(2)} \Rightarrow 2 = e^{k(2)}$

$\Rightarrow \ln 2 = 2k \ln e = 2k$

$\Rightarrow \frac{\ln 2}{2} = k$

$\rightarrow y = 2e^{\left(\frac{\ln 2}{2}\right)t} \Rightarrow y(3) = 2e^{\frac{\ln 2}{2}(3)} = 5.657$

Ex 3: 10g plutonium isotope $^{239}Pu$

Half life is 24,100 years $\quad$ How long to decay to 1g

$y = Ce^{kt}$ $\quad$ find k $\quad \frac{1}{2} = e^{k(24,100)}$

$\Rightarrow \ln\left(\frac{1}{2}\right) = k\,24,100 \Rightarrow k = \frac{\ln\left(\frac{1}{2}\right)}{24,100}$

$1 = 10e^{\frac{\ln\frac{1}{2}}{24,100}(t)} \rightarrow \frac{1}{10} = e^{\frac{\ln\frac{1}{2}}{24100}t}$

$\ln\frac{1}{10} = \frac{\ln\frac{1}{2}}{24100}t \Rightarrow \frac{\ln\frac{1}{10}}{\frac{\ln\frac{1}{2}}{24100}} = t$ $\Rightarrow$ It takes 80,058.467 years for 10g to become 1g.

# 6.3 Seperation of Variables

Warm up:

$\frac{dy}{dx} = xy \Rightarrow \int \frac{1}{y}\,dy = \int x\,dx \Rightarrow \ln|y| = \frac{x^2}{2} + C$

$y = e^{\frac{x^2}{2} + C} \Rightarrow y = Ce^{\frac{x^2}{2}} \Rightarrow \boxed{\frac{1}{2} = C} \quad \boxed{y = \frac{1}{2}e^{\frac{x^2}{2}}}$

Always +C !!!

## Ex 1:

$(x^2+4)\frac{dy}{dx} = xy$

$\int \frac{1}{y}\,dy = \int \frac{x}{x^2+4}\,dx \qquad u = x^2+4 \quad du = 2x\,dx$

$\Rightarrow \ln|y| = \frac{1}{2}\ln|x^2+4| + C$

$e^{(\ln|y|} = e^{(\ln\sqrt{x^2+4} + C)} \Rightarrow \boxed{y = C\sqrt{x^2+4}}$

## Ex 2:

$(1,3)$; $\frac{dy}{dx} = \frac{y}{x^2}$

$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{1}{x^2}\,dx \Rightarrow \ln|y| = \int x^{-2}\,dx + C$

$\Rightarrow \ln|y| = -1x^{-1} + C = \frac{-1}{x} + C$

$y = Ce^{\frac{-1}{x}} \Rightarrow 3 = Ce^{-\frac{1}{1}} \Rightarrow 3 = \frac{C}{e}$

$\Rightarrow 3e = C \Rightarrow y = 3e(e^{-\frac{1}{x}}) = \boxed{y = 3e^{1-\frac{1}{x}}}$

## Ex 3:

$\frac{dy}{dx} = x^4(y-2)$ AP FRQ

+1pt

$\int \frac{1}{y-2}\,dy = \int x^4\,dx \Rightarrow \ln|y-2| = \frac{x^5}{5} + C$

+1pt +1pt +1pt

Initial condition: $(0,0)$

$y-2 = Ce^{\frac{x^5}{5}} \Rightarrow y = Ce^{\frac{x^5}{5}} + 2 \Rightarrow 0 = Ce^0 + 2$

$C = -2$ +1pt $\Rightarrow \boxed{y = -2e^{\frac{x^5}{5}} + 2}$ +1pt

6/6

# Chapter 5 & 6 Quiz Review

Always round to 3 decimal places: 0.000

(4) $\int_{\ln 4}^{\ln 7} e^{-x}\,dx$

$u = -x$
$du = -dx$
$u' = -1$

$\rightarrow -\int e^{u}\,du \rightarrow \left[-e^{-x}\right]_{\ln 4}^{\ln 7}$

$\rightarrow \left[\frac{-1}{e^{x}}\right]_{\ln 4}^{\ln 7} \rightarrow \left[\frac{-1}{e^{\ln 7}} - \frac{-1}{e^{\ln 4}}\right] \rightarrow -\frac{1}{7} + \frac{1}{4}$

$\rightarrow \frac{-4}{28} + \frac{7}{28} \rightarrow \boxed{\frac{3}{28}}$

(2) $f(x)$ & $g(x)$ are inverse if $f(x) = 4x^5 + 4x + 2$

$f'(x) = 20x^4 + 4$

find $g'(10)$:

| | x | y |
|---|---|---|
| g | 10 | 1 |
| f | 1 | 10 |

$10 = 4x^5 + 4x + 2 \rightarrow 0 = 4x^5 + 4x - 8$

$\rightarrow 0 = 4(x^5 + x - 2) \qquad \frac{p}{q} = \frac{\pm 1, \pm 2}{\pm 1}$

$$\begin{array}{r|rrrrrr} 1 & 1 & 0 & 0 & 0 & 1 & -2 \\ & & 1 & 1 & 1 & 1 & 2 \\ \hline & 1 & 1 & 1 & 1 & 2 & 0 \end{array}$$

$\rightarrow g'(10) = \frac{1}{f'(g(10))} \rightarrow \frac{1}{f'(1)} = \frac{1}{20+4} \rightarrow \boxed{\frac{1}{24}}$

(3) $y' = \frac{8y}{9x} \rightarrow y' = \frac{8}{9} \cdot \frac{y}{x} \rightarrow \int \frac{1}{y}\,dy = \int \frac{8}{9} \frac{1}{x}\,dx$

$\rightarrow \ln|y| = \frac{8}{9}\ln|x| + C \rightarrow \ln y = \ln x^{8/9} + C$

$y = Cx^{8/9} \rightarrow @(7, 9) \rightarrow 9 = C(7)^{8/9}$

$\rightarrow C = \frac{9}{7^{8/9}} \rightarrow \boxed{y = \left(\frac{9}{7^{8/9}}\right)x^{8/9}}$

(12) $\int \frac{2x-4}{x^2-6x+45}\,dx$ $\quad u=x^2-6x+45 \quad du=(2x-6)dx$

$\int \frac{2x-6}{x^2-6x+45}\,dx + \int \frac{2}{x^2-6x+45}\,dx$

$\to \int \frac{1}{u}\,du + 2\int \frac{1}{x^2-6x+45}\,dx$ $\quad u=x-3 \quad a=6$

$\to \ln|x^2-6x+45| + 2\int \frac{1}{(x-3)^2+36} + C$

$\to \boxed{\ln|x^2-6x+45| + 2\left(\frac{1}{6}\right)\arctan\left(\frac{x-3}{6}\right) + C}$

(5) $\int e^{4x}\sec(e^{4x})\tan(e^{4x})\,dx$

$u=e^{4x}$
$du=4e^{4x}$ $\quad \to \frac{1}{4}\int \sec u \tan u\,du$

$\to \frac{1}{4}\sec u + C \to \boxed{\frac{1}{4}\sec(e^{4x}) + C}$

(13) $\int \frac{1}{x\sqrt{x^4-4}}\,dx$ $\quad u=x^2 \quad du=2x\,dx \quad a=2$

$\to \frac{1}{2}\int \frac{2x}{x(x)\sqrt{x^4-4}}\,dx \to \frac{1}{2}\int \frac{1}{u\sqrt{u^2-a^2}}\,du$

$\frac{1}{2}\left(\frac{1}{2}\right)\sec^{-1}\frac{|x^2|}{2} + C \to \boxed{\frac{1}{4}\sec^{-1}\frac{|x^2|}{2} + C}$

(6) $f'(x)$ if $f(x)=(x^8)(2^{7x})$ $\quad \frac{d}{dx}[a^u] \to \ln a\, a^u u'$

$\to x^8(\ln 2\,(2^{7x})\,7) + 8x^7(2^{7x})$

(10) $\int \frac{x}{x^4+9}\,dx$ $\quad \to u=x^2 \quad a=3 \quad du=2x\,dx$

$\frac{1}{2}\int \frac{1}{u^2+a^2}\,du \to \boxed{\frac{1}{2}\left(\frac{1}{3}\arctan^{-1}\frac{x^2}{3}\right) + C}$

(11) $\int_0^{1/6} \frac{3}{\sqrt{1-9x^2}}\,dx$ $\quad u=3x \quad du=3\,dx \quad a=1$

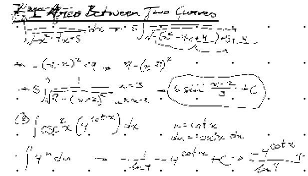
$\to \left[\arcsin\frac{3x}{1}\right]_0^{1/6} \to \left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)\right] \to \frac{\pi}{6} - 0 \to \boxed{\frac{\pi}{6}}$

(7) $\int 2^{7x}\,dx$ $\quad \int a^u = \frac{1}{\ln a} a^u + C$

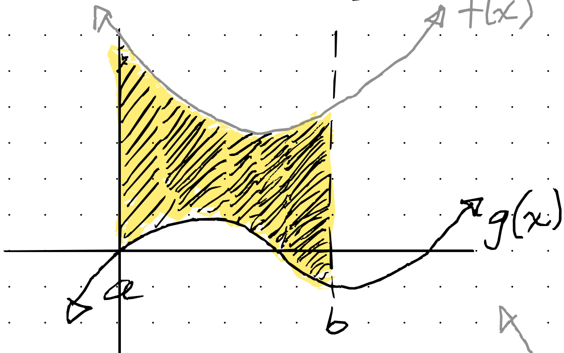
$u = 7x$

$du = 7\,dx$ $\quad \frac{1}{7}\int a^u\,du \Rightarrow \frac{1}{7}\left(\frac{1}{\ln 2}\, 2^{7x}\right) + C$

# 7.1 Area Between Two Curves

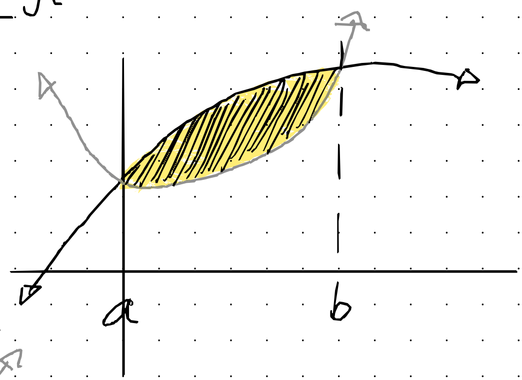


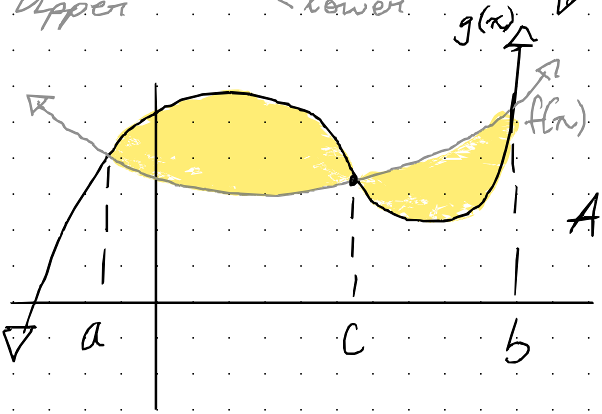
Area between 2 curves:




$$A=\int_a^b\left(f(x)-g(x)\right)\,dx$$

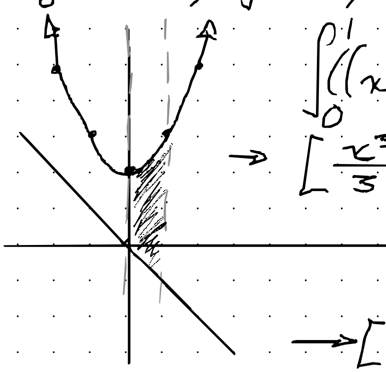
upper lower




$$A=\int_a^c\left(g(x)-f(x)\right)dx+\int_c^b\left(f(x)-g(x)\right)dx$$

Ex 1: $y=x^2+2$, $y=-x$, $x=0$, $x=1$



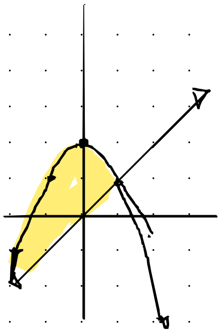
$$\int_0^1\left((x^2+2)-(-x)\right)dx$$

$$\rightarrow\left[\frac{x^3}{3}+2x+\frac{x^2}{2}\right]_0^1$$

$$\left[\left(\frac{1}{3}+2+\frac{1}{2}\right)-0\right]$$

$$\rightarrow\left[\frac{2}{6}+\frac{12}{6}+\frac{3}{6}\right]\rightarrow\boxed{\frac{17}{6}}$$

Ex 2: $f(x) = 2 - x^2 \quad g(x) = x$



$[-2, 1] \quad f = g$

$2 - x^2 = x$

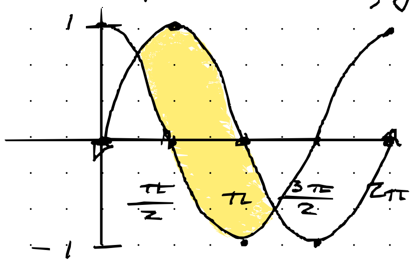
$0 = x^2 + x - 2 \to 0 = (x+2)(x-1)$

$A = \int_{-2}^{1} \left( (2 - x^2) - (x) \right) dx$ $\qquad \underline{x = -2}$

$\to \boxed{\frac{9}{2}}$ or $4.5$ $\qquad \underline{x = 1}$

If $[a,b]$ is missing, find the missing x-values through solving for points of intersection

Ex 3: $f(x) = \sin x, \; g(x) = \cos(x)$




$A = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x) \, dx$

$\to \boxed{2.828}$

Ex 4: $f(x) = 3x^3 - x^2 - 10x \quad g(x) = -x^2 + 2x$

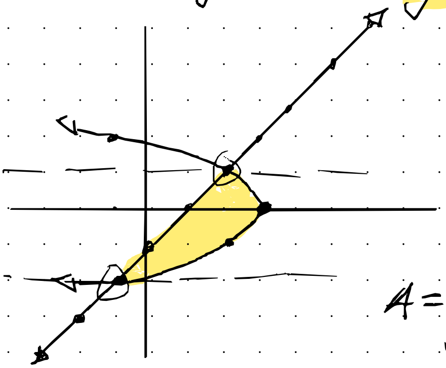
$A = \int_{-2}^{0} (f(x) - g(x)) dx + \int_{0}^{2} (g(x) - f(x)) dx$

$12 \quad + \quad 12 \quad \to \boxed{24}$

Ex5: $x = 3 - y^2 \quad x = y + 1$

$\hookrightarrow y = x - 1$

$\hookrightarrow y^2 = 3 - x \Rightarrow y = \pm\sqrt{3-x}$



$3 - y^2 = y + 1 \Rightarrow 0 = y^2 + y - 2$

$0 = (y+2)(y-1)$

$y = -2 \quad y = 1$

$$A = \int_{-2}^{1} \left( (3 - y^2) - (y + 1) \right) dy$$

$\longrightarrow \boxed{9/2}$ or $4.5$

Note:

The "a" & "b" limits come from "y" axis in horizontal problems, not the "x". Use "dy".

Solve the points of intersections for "y".

Reimagine the integral in terms of "x" to think of it easier. The upper function is to the right.

# 7.2 Disk & Washer Method

7.2 Disk and Washer Method

$x^2+8x+12 = -x-2 \to 0 = x^2+4x+14$ $x=-3$ $x=-1$ $-2$

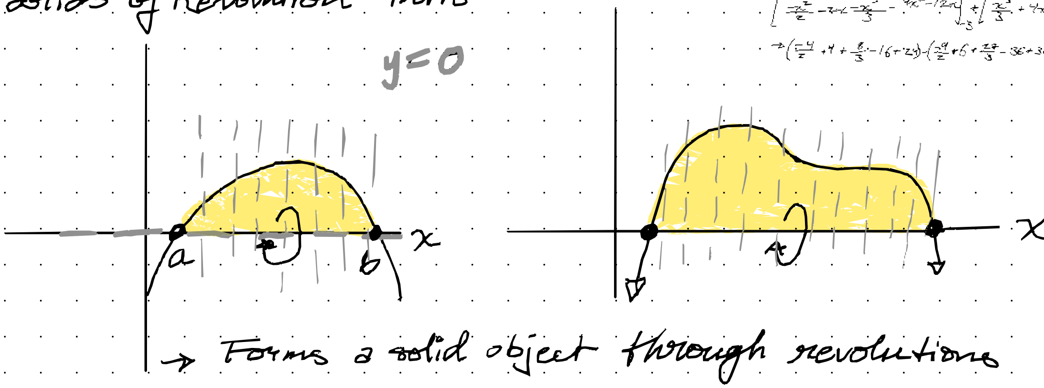
$A = \int_{-3}^{-2}((-x-2)-(x^2+8x+12))\,dx + \int_{-2}^{-1}((x^2+8x+12)+x+2)\,dx$

$\int_{-3}^{-2} -x-2-x^2-8x-12\,dx$ $\int_{-2}^{-1} x^2+8x+12+x+2$

$\left[-\frac{x^2}{2}-2x-\frac{x^3}{3}-4x^2-12x\right]_{-3}^{-2} + \left[\frac{x^3}{3}+4x^2+12x+\frac{x^2}{2}+2x\right]_{-2}^{-1}$

$\to \left(\frac{-4}{2}+4+\frac{8}{3}-16+24\right)-\left(\frac{-9}{2}+6+\frac{27}{3}-36+36\right)$

## Solids of Revolution Intro

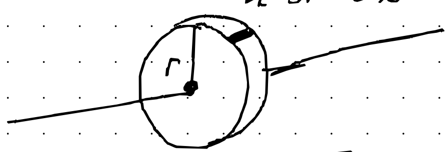



→ Forms a solid object through revolutions

→ Then slice into discs to integrate

$$V\ \text{Cylinder/disc} = \underbrace{\boxed{\pi r^2}}_{}\!\!{}^{=\text{Area of base}}\, h \to \Delta x \text{ or } \frac{d}{dx}$$

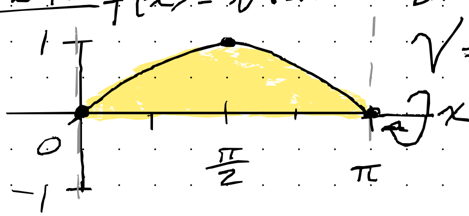
$$\int_a^b \pi r^2\,dx$$




$$V = \pi\int_a^b (f(x))^2\,dx$$

$$\to \boxed{\pi\int_a^b R(x)^2\,dx}$$ Volume of revolvable solid

**Ex 1:** $f(x) = \sqrt{\sin x}$ and $0 \le x \le \pi$




$$V = \pi\int_0^{\pi} \sqrt{\sin x}^{\,2}\,dx$$

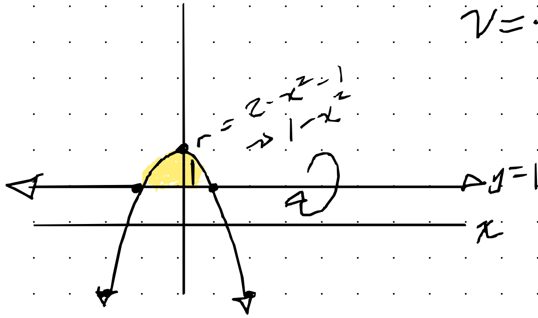
$$\to \pi\int_0^{\pi} \sin x\,dx \approx \boxed{6.283}$$

or $2\pi$

Ex 2: $f(x) = 2 - x^2$, $g(x) = 1$, $y = 1$ (revolve about)

$$V = \pi \int_{-1}^{1} (1 - x^2)^2 \, dx$$

$$\approx 3.351$$



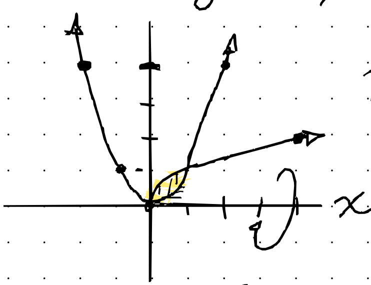

Washer method: $\pi \int_a^b \left( R(x)^2 - r(x)^2 \right) dx$

$R(x)$ = Outermost radius

$r(x)$ = Radius of hole in center

With vertical method: $V = \pi \int_c^d \left( [R(y)]^2 - [r(y)]^2 \right) dy$

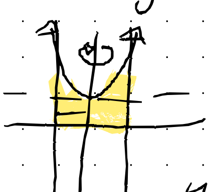
Ex 3: $y = \sqrt{x}$, $y = x^2$, about the x-axis




$$V = \pi \int_0^1 \left( \overset{R}{(\sqrt{x})^2} - \overset{r}{(x^2)^2} \right) dx$$

$$V = \pi \int_0^1 (x - x^4) \, dx \approx \boxed{0.942}$$

Ex 4: $y = x^2 + 1$, $y = 0$, $x = 0$, $x = 1$ about y-axis



$$V = \pi \int_0^1 (1)^2 \, dy + \pi \int_1^2 \left( [1]^2 - [\sqrt{y-1}]^2 \right) dy$$

$y - 1 = x^2 \rightarrow x = \pm\sqrt{y-1}$ $\approx 4.712$

only use +, taking integral

# Cross Sections:

Volumes of Solids with Known Cross Sections

1. Taken perpendicular to $x$-axis:

$$V = \int_a^b A(x)\,dx$$

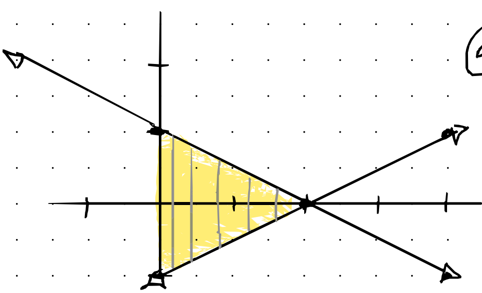
2. Taken perpendicular to $y$-axis:

$$V = \int_c^d A(y)\,dy$$

Ex 6: Base is bounded by the lines

$f(x) = 1 - \frac{x}{2}$, $g(x) = -1 + \frac{x}{2}$, and $x = 0$

Find the volume. The cross sections are perpendicular to the $x$-axis. (a) squares (b) semi-circles



(a) $V = \int_0^2 \left(\left(1 - \frac{x}{2}\right) - \left(-1 + \frac{x}{2}\right)\right)^2 dx$

$\approx 2.667 = \boxed{\frac{8}{3}}$

(b) Semi-circle $A = \frac{1}{2}\pi r^2 = \frac{\pi}{2} r^2$

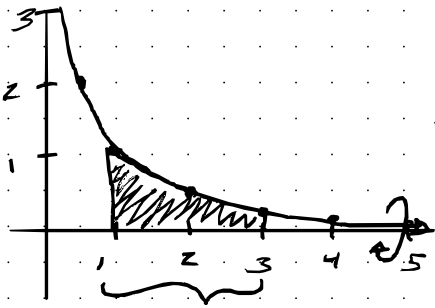
$$V = \int_0^2 \left(\frac{\pi}{2}\left(\frac{\left(1 - x/2\right) - \left(-1 + \frac{x}{2}\right)}{2}\right)^2\right) dx$$

$$= \frac{\pi}{8}\int_0^2 \left(\left(1 - \frac{x}{2}\right) - \left(-1 + \frac{x}{2}\right)\right)^2 dx$$

$$\approx \boxed{1.041}$$

# Ch 7 Practice

$y=\frac{1}{x}$, x-axis, $x=1, 3$



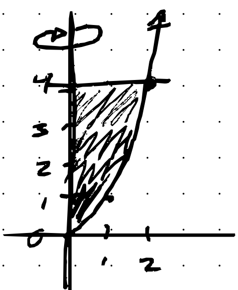

$$V=\pi\int_1^3 \left(\frac{1}{x}\right)^2 dx$$

$$\rightarrow V=\pi\int_1^3 x^{-2}\, dx$$

$$\rightarrow V=\pi \left.\frac{x^{-1}}{-1}\right|_1^3 \rightarrow \pi \left.\frac{-1}{x}\right|_1^3$$

$$\rightarrow V=\pi\left[-\frac{1}{3}+\frac{1}{1}\right] \rightarrow \pi\,\frac{2}{3} \text{ or } \boxed{\frac{2\pi}{3}}$$

$y=x^2$, $x=0$, $y=4$, y-axis




$$x=\pm\sqrt{y}$$

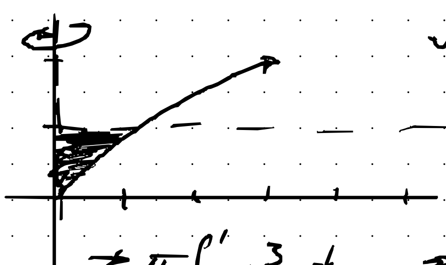
$$V=\pi\int_0^4 R^2(y)\, dy \rightarrow V=\pi\int_0^4 \left(\sqrt{y}\right)^2 dy$$

$$\rightarrow \pi\int_0^4 y\, dy \rightarrow \pi \left.\frac{y^2}{2}\right|_0^4$$

$$\rightarrow \pi\left[\frac{16}{2}-\frac{0}{2}\right] \rightarrow \boxed{8\pi}$$

$$y=x^{2/3}$$
$$y^3=x^2$$
$$x=\pm\sqrt{y^3}$$

$y=x^{2/3}$, $x=0$, $y=1$, y-axis



$$V=\pi\int_0^1 R^2(y)\, dy \rightarrow$$

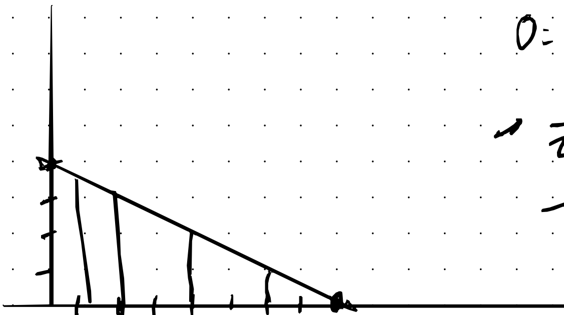
$$V=\pi\int_0^1 \sqrt{y^3}^{\,2}\, dy$$

$$\rightarrow \pi\int_0^1 y^3\, dy \rightarrow \pi \left.\frac{y^4}{4}\right|_0^1 \rightarrow \pi\left[\frac{1}{4}-\frac{0}{4}\right]$$

$$\rightarrow \boxed{\frac{\pi}{4}}$$

$x=0$, $y=0$, $y=4-\frac{1}{2}x$ use semicircles perpendicular to the x-axis.

$A_{\circ} = \frac{1}{2}\pi r^2$



$0 = 4 - \frac{1}{2}x$

$\frac{1}{2}x = 4$

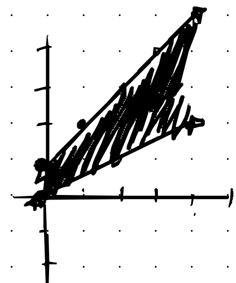
$x = 8$

$V = \frac{\pi}{2}\int_0^8 \left(2 - \frac{1}{4}x\right)^2 dx$

$\approx 16.755$

$\left(\frac{1}{2}\left(4 - \frac{1}{2}x\right)\right)^2$

7) $\int_0^4 \left[(x+1) - \frac{x}{2}\right] dx \to$



31) $f(x) = x(x^2-3x+3)$ $g(x) = x^2$

$x^3 - 3x^2 + 3x = x^2 \to 0 = x^3 - 4x^2 + 3x$

$x = 0, 1$

$A = \int_0^1 \left((x^3 - 3x^2 + 3x) - (x^2)\right) dx$

$\int_0^1 (x^3 - 3x^2 + 3x - x^2) dx \to \int_0^1 (x^3 - 4x^2 + 3x) dx$

$\to \left[\frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2}\right]_0^1 \to \left[\left(\frac{1}{4} - \frac{4}{3} + \frac{3}{2}\right) - (0 - 0 + 0)\right]$

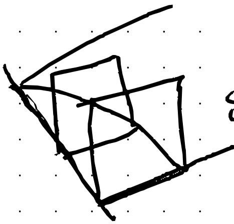
$\to \left(\frac{3}{12} - \frac{16}{12} + \frac{18}{12}\right) \to \frac{3}{12} + \frac{2}{12} = \frac{5}{12}$

$\int_1^3 \left(x^2 - (x^3 - 3x^2 + 3x)\right) dx \to \int_1^3 (-x^3 + 4x^2 - 3x) dx$

$\left[-\frac{x^4}{4} + \frac{4x^3}{3} - \frac{3x^2}{2}\right]_1^3 \to \left[\left(\frac{81}{4} + \frac{108}{3} - \frac{27}{2}\right) - \left(\frac{-1}{4} + \frac{4}{3} - \frac{3}{2}\right)\right]$

$\frac{81}{4} + \frac{108}{3} - \frac{27}{2} + \frac{1}{4} - \frac{4}{3} + \frac{3}{2}$ $+ \frac{5}{12} \to \frac{811}{12}$

$y = \sqrt{x}$, $x = 4$, $y = 0$, squares with base perpendicular to x-axis.



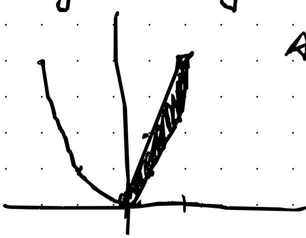
$$V = \int_0^4 \sqrt{x}^{\,2}\, dx$$

$$S_{\square B} = \sqrt{x}$$

$$V = \int_0^4 x\,dx \to \frac{x^2}{2}\Big|_0^4 \to \left[\frac{16}{2} - \frac{0}{2}\right]$$

$$\to \boxed{8}$$

$y = x^2$ $\quad y = 2x$ $\quad (0,2)$

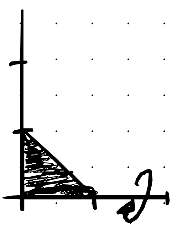


$$A = \int_0^2 (2x - x^2)\,dx$$

$$\left[x^2 - \frac{x^3}{3}\right]_0^2 \to \left[\left(4 - \frac{8}{3}\right) - \cancel{(0-0)}\right]$$

$$4 - \frac{8}{3} \to \frac{12}{3} - \frac{8}{3} \to \boxed{\frac{4}{3}} \quad \text{or } 1.\overline{33}$$

① $y = -x + 1$



$$V = \pi \int_0^1 \left((-x+1)^2 - 0^2\right) dx$$

$$(-x+1)$$

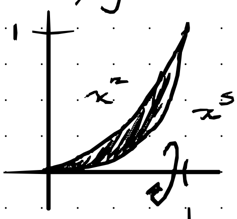
$$\to \pi \int_0^1 (x^2 - 2x + 1)\, dx$$

$$\pi\left[\frac{x^3}{3} - x^2 + x\right]_0^1 \to \pi\left[\left(\frac{1}{3}\right) - (0)\right]$$

$$\to \frac{\pi}{3}$$

$(0,1)$

⑤ $y = x^2$, $y = x^5$



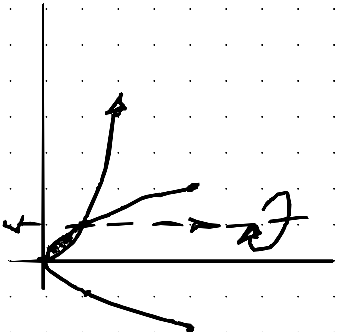

$$V = \pi \int_0^1 \left((x^2)^2\right) dx - \pi \int_0^1 \left((x^5)^2\right) dx$$

$$\to \pi\left[\frac{x^5}{5}\right]_0^1 - \pi\left[\frac{x^{11}}{11}\right]_0^1$$

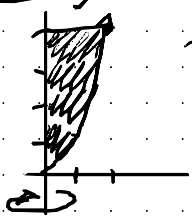
$$\to \frac{\pi}{5} - \frac{\pi}{11} \to \frac{22\pi}{110} - \frac{10\pi}{110} \to \frac{12\pi}{110}$$

$$\frac{6\pi}{55}$$

$y = x^2$, $x = y^2$, about $y = 1$ $\boxed{dx}$



7 $y = x^2$ $\pm\sqrt{y} = x$ $(0, 4)$

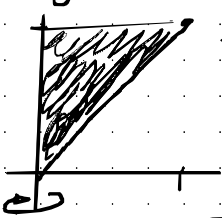


$V = \pi \int_0^4 (\sqrt{y}^2 - 0^2)\, dy$ $\pi \int_0^4 y \, dy$

$\to \pi \left[ \frac{y^2}{2} \right]_0^4$ $\pi \left[ \frac{16}{2} - 0 \right] \to \boxed{8\pi}$

9 $y = x^{2/3} \to \sqrt[3]{x^2} \to y^3 = x^2 \to x = \sqrt{y^3}$ or $y^{3/2}$

$(0, 1)$

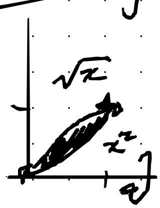


$V = \pi \int_0^1 (\sqrt{y^3}^2 - 0^2)\, dy$

$\to \pi \int_0^1 y^3 \, dy \to \pi \left[ \frac{y^4}{4} \right]_0^1 \to \pi \left[ \left( \frac{1}{4} \right) - 0 \right]$

$\to \boxed{\frac{\pi}{4}}$

Ex 3: $y = x^2$ $y = \sqrt{x}$

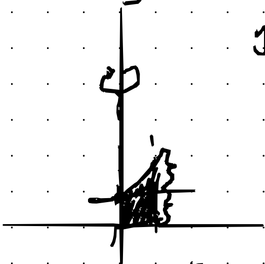



$V = \pi \int_0^1 (\sqrt{x}^2 - (x^2)^2)\, dx$

$\to \pi \int_0^1 (x - x^4)\, dx \to \pi \left[ \frac{x^2}{2} - \frac{x^5}{5} \right]_0^1$

$\to \pi \left[ \left( \frac{1}{2} - \frac{1}{5} \right) - 0 \right] \to \pi \left[ \frac{5}{10} - \frac{2}{10} \right] \to \frac{3\pi}{10}$

Ex 4: $y = x^2+1,\ y=0,\ x=0,\ x=1$



$y-1 = x^2 \to \sqrt{y-1} = x$

$V = \pi\int_1^2 \left(1^2 - (\sqrt{y-1})^2\right)dy + \pi\int_0^1 (1^2)\,dy$

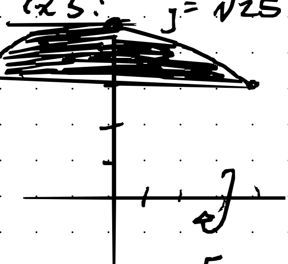
$\to \pi\int_1^2 \left(1-(y-1)\right)dy + \pi\int_0^1 1\,dy$

$\to \pi\left[1y - \frac{y^2}{2} + y\right]_1^2 + \pi\left[y\right]_0^1 \to \pi\left[(4-2)-\left(2-\frac{1}{2}\right)\right] + \pi[1]$

$2y - \frac{y^2}{2}$ $\qquad \to \pi\left[2 - \frac{3}{2}\right] + \pi$

$\to \frac{\pi}{2} + \pi \to \boxed{\frac{3\pi}{2}}$

Ex 5: $y = \sqrt{25-x^2}$ $\quad y = 3$ $\quad$ G x-axis



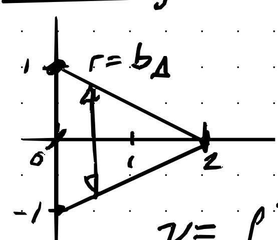
$V = \pi\int_{-4}^{4}\left(\sqrt{25-x^2}^{\,2} - 3^2\right)dx$

$\to \pi\int_{-4}^{4}\left(25-x^2-9\right)dx \to \pi\int_{-4}^{4}\left(16-x^2\right)dx$

$\to \pi\left[16x - \frac{x^3}{3}\right]_{-4}^{4}$ $\quad \pi\left[\left(64-\frac{64}{3}\right)+\left(+64+\left(\frac{-64}{3}\right)\right)\right]$

$\to \pi\left[\left(\frac{128}{3}\right)+\frac{128}{3}\right] \to \frac{256\pi}{3}$

Ex 6: $f(x) = 1-\frac{x}{2}$ $\quad g(x) = -1+\frac{x}{2}$ $\quad x = 0$ $\qquad (0,2)$




$A_\Delta = \frac{\sqrt{3}}{4}b^2$

$b = 1-\frac{x}{2} - \left(-1+\frac{x}{2}\right)$

$\to 2 - x$ $\qquad A(x) = \frac{\sqrt{3}}{4}(2-x)^2$

$V = \int_0^2 \left(\frac{\sqrt{3}}{4}(2-x)^2\right)dx$

# Practice for Mock Exam

① $(2x^2+5)^7$ $\quad u=2x^2+5 \rightarrow u^7$

$\rightarrow 7(2x^2+5)^6(4x) \rightarrow 28x(2x^2+5)^6$

$u=3$

② $\int \frac{1}{3x+12}\,dx \quad \frac{1}{3}\int \frac{1}{u}\,du$

$\frac{1}{3} \ln \frac{1}{3x+12}$

③ $\frac{5-x}{x^3+2} \rightarrow \frac{-1}{3x^2}$ $\quad \frac{(x^3+2)(-1)-(5-x)(3x^2)}{(x^3+2)^2}$

$-x^3-$

$+3x^3 \rightarrow 2x^3$

④ $A_{\text{trapezoid}} = \frac{1}{2}(h)(b_1+b_2)$

$A = \frac{1}{2}\left(\frac{1}{2}\right)(80) + \frac{1}{2}\left(\frac{3}{2}\right)(60+40) + \frac{1}{2}(1)(40+30)$

$\frac{1}{4}\,80 + \frac{3}{4}(100) + \frac{1}{2}\,70$

$20 + 75 + 35 \rightarrow 130$

⑤ $\sin(x^2+\pi)$ $\quad f'(\sqrt{2\pi})$

$\rightarrow u = x^2+\pi \rightarrow \cos(x^2+\pi)(2x)$

$\rightarrow 2x$

$\rightarrow \cos(2\pi+\pi)(2\sqrt{2\pi})$

$\rightarrow \cos(3\pi)(2\sqrt{2\pi})$

$-1(2\sqrt{2\pi})$

(6) $3x^2 - x^3 \to \cancel{f' = 6x - 3x^2}$

$\cancel{f'(1) = 6 - 3 \to 3}$ $\to -42/2$

$\cancel{f'(5) = 30 - 75 \to -45}$ $\to -21$

$f(1) = 3 - 1 \to 2$ $\qquad \frac{-50-2}{5-1} \to \frac{-52}{4}$

$f(5) = 75 - 125 \to -50 \to$

$\to -26 \to -13$

(8) $f(x) = e^{x/3} \to f'(x) = e^{x/3}\left(\frac{1}{3}\right)$

$\frac{e^{x/3}}{3}$ of $(3\ln 4) \to \frac{e^{\cancel{3}\ln 4/\cancel{3}}}{3} \to \frac{4}{3}$

$y - 4 = \frac{4}{3}(x - 3\ln 4)$

(10) $\int_0^2 (x^3+1)^{1/2} x^2\, dx \to \int_0^2 \sqrt{x^3+1}\; x^2\, dx$

(11) $x^2 + xy - 3y = 3 \to x^2 + xy - 3y - 3 = 0$

$2x + \left(x\frac{dy}{dx} + 1(y)\right) - 3\frac{dy}{dx} = 0 \qquad \frac{dy}{dx} \to 2x + y - 3\frac{dy}{dx} + x\frac{dy}{dx} = 0$

$x\frac{dy}{dx} - 3\frac{dy}{dx} = -2x - y \to \frac{dy}{dx}(x-3) = -2x - y \to \frac{dy}{dx} = \frac{-2x-y}{x-3}$

$(2,1) \to \frac{-2(2)-1}{2-3} \to \frac{-4-1}{-1} \to \frac{-5}{-1} \to \boxed{5}$

(12) $w'(t) = -2t \to -2(3) = -6$

(14) $y = Ce^{kt}$

(16) $f'(x) = 12x - 6x^2$ $\qquad f''(x) = 12 - 12x$ $\qquad 120$ $\qquad 12 - 6$

$4\ 6$ up

$6 - 1.5 \to$ $\qquad \frac{3}{2}$ $\qquad 12 - 18$

up

$-12e^{-t}(\sin t) + 12e^{-t}(\cos t) = 0$ | $\cos t - \sin t = 0$

$12e^{-t}(-\sin t + \cos t) = 0$

$\cancel{12e^{-t} = 0}$ $\quad (\sin + \cos t) = 0$

(18) $(\tan 5x)\sec(5x) - 1 = F(x)$

$-2\sin x \quad \frac{\pi}{2} \to -2\sin\frac{\pi}{2} \to -2$

$2\cos\frac{\pi}{2} + 1 \to \left(\frac{\pi}{2}, 1\right)$

$y - 1 = -2\left(x - \frac{\pi}{2}\right) \to y_? = -2\left(x - \frac{\pi}{2}\right) + 1$

$y_?(1.5) = -2\left(1.5 - \frac{\pi}{2}\right) + 1 \to -3 + \pi + 1$

$\to \pi - 2$

$\lim_{x \to 3} \frac{\ln x - \cancel{\ln 3}}{x - 3} \to \frac{\frac{1}{x}}{1} \to \frac{1}{3}$

$\frac{1}{2y}dy = \frac{1}{2x+1}dx \to \frac{1}{2}\ln\frac{1}{2x+1}$

$u' = 2$

$g(x) = -2x \qquad f(x) = \frac{1}{2}x$

$-2x \qquad \to \frac{1}{2}$

$-2x\left(\frac{1}{2}\right) + (2)\left(\frac{1}{2}x\right) \to \quad -x - x$

$\to -2x \to \quad -2(2) \to 4$

$g(x) = -2x + 5 \qquad f(x) = \frac{1}{2}x + 2$

$f'(x) = \frac{1}{2} \qquad g'(x) = -2$

$(-2x+5)\left(\frac{1}{2}\right) + \left(\frac{1}{2}x + 2\right)(-2) \quad @\,2 \to (-4+5)\frac{1}{2} + (3)(-2)$

$\to \frac{1}{2} + -6 \to -\frac{11}{2}$

$A = \frac{1}{2}s^2$ $\quad s = \sqrt{32}$ $\quad h = \sqrt{64}$

$\frac{dA}{dx} = +12\, cm^2/s$ $\quad$ find $\frac{ds}{dx}$ $\quad A = \frac{1}{2} 32 = 16$ $\quad \frac{12}{8} \to 6$

$\frac{ds}{dx} = \cdot \to \frac{ds}{dx} = 12\, cm/s$

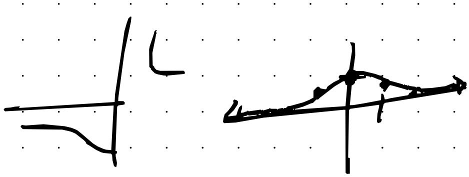
Yuck

$12^2 + 12^2 = c^2$

$144 + 144 = c^2$

$288 = \sqrt{288}$

$\frac{1}{x^2+1} \quad \frac{1}{x^3+1} \quad \frac{1}{e^x-1} \quad \frac{1}{e^x+1}$

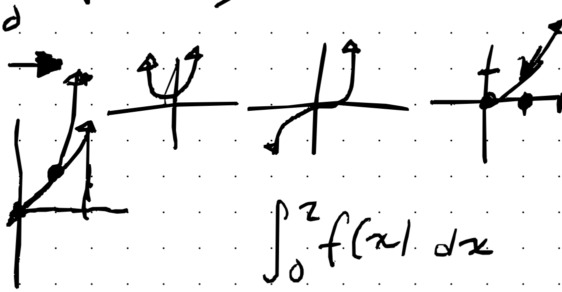


$x^{3\frac{1}{2}} \to x^{3/2}$

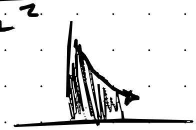
$\int_0^2 (x^3+1)^{1/2} x^2\, dx \to \int_0^2 \sqrt{x^3+1}\, x^2\, dx$

$2\sqrt{2} \to 2(1.?)$

$\to 2.?$



$\int_0^2 f(x)\, dx$ $\quad \frac{2(n+1)(3n+2)}{n^2}$



$2(?) \quad \frac{\to 4}{0} \to \infty$

$4(5) \to \underline{20} \to 20$

$\frac{6(8)}{4} \to \frac{48}{4} \to 12 \to 12$

$\int 8x\sqrt{40-2x^2}\,dx$ $\qquad u = 40-2x^2$

$du = -4x\,dx$

$\int 8\cancel{x}\sqrt{u}\,\dfrac{du}{-4\cancel{x}}$ $\qquad dx = \dfrac{du}{-4x}$

$-2\int \sqrt{u}\,du \to u^{1/2} \to -2\left[\dfrac{u^{3/2}}{3/2}\right] + C$

$\to \dfrac{-4\left[\sqrt{40-2x^2}\right]^3}{3} + C$

$\dfrac{1}{b-a}\int_a^b f(x)\,dx$ = average value

$\dfrac{f(b)-f(a)}{b-a}$ $[a,b]$

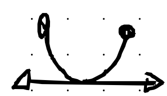
of $f(x) = x^2+4x-5$ $\qquad [1,3]$

$^a f(1) = 1^2+4(1)-5 \to 0$

$^b f(3) = 3^2+4(3)-5 \to 9+12-5 \to 16$

$\dfrac{16-0}{3-1} \to \boxed{8}$

$f(x) = x^3-4$ $\qquad [2,5]$



$f(2) = 2^3-4 \to 8-4 \to 4$

$f(5) = 125-4 \to 121$

$\dfrac{121-4}{5-2} \to \dfrac{117}{3} = 39$

$u = 2+x^4$ $\qquad du = 4x^3\,dx$

$\int \dfrac{x^3}{(2+x^4)^2}\,dx$ $\qquad dx = \dfrac{du}{4x^3}$

$u^{-2} \to \dfrac{u^{-1}}{-1} \to -\dfrac{1}{u}$

$\int \dfrac{\cancel{x^3}}{(u)^2}\,\dfrac{du}{4\cancel{x^3}} \to \dfrac{1}{4}\int \dfrac{1}{u^2}\,du \to -\dfrac{1}{4}\left(\dfrac{1}{2+x^4}\right) + C$

# Review of Mock Exam: MCQ 1

Memorize trapazoidal rule

$$\int_a^b f(x)\,dx = \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + \ldots 2f(x_{n-1})\right]$$

[ Practice implicit differentiation

Find tangent at certain x-val on function

Integrating with u-sub

$$\frac{f(x) = x^3}{f'(x) = 3x^2}$$

$g'(2)$

| $g$ | $g(x)$ |
|---|---|
| 2 | 4 |

$$\frac{1}{3(2)^2} = \frac{1}{12}$$